Exercise sheet No. 3

Deadline: 04/11/2019.
You should hand only exercises 0.1 and 0.2.

Exercise 0.1 (5 points)**.** Let M be an orientable closed connected non-empty manifold. Prove that any closed connected manifold N which covers M is orientable.

Exercise 0.2 (5 points)**.** Using the homological definition of orientation, prove that $\mathbb{RP}^2$ is a non-orientable manifold. You could also help yourself by drawing a nice picture.

Exercise 0.3. Let M be a closed connected non-empty n-manifold. We construct the following space

$$\tilde{M} \coloneqq \{\mu_x \,|\, x \in M \text{ and } \mu_x \text{ is a local orientation of } M \text{ at } x\} \ .$$

The aim of this exercise is to prove that $\tilde{M}$ is a two-sheeted covering space of M, where the projection is given by $\mu_x \mapsto x$. Prove that

(i) The map $p\colon \mu_x \mapsto x$ is a $2:1$ surjection.

Given an open ball $B \subset \mathbb{R}^n \subset M$ of finite radius and a generator $\mu_B \in H_n(M, M \setminus B; \mathbb{Z})$, we set $U(\mu_B)$ to be the set of all $\mu_x \in \tilde{M}$, where $x \in B$ and μ_x is the image of μ_B under the map $H_n(M, M \setminus B; \mathbb{Z}) \to H_n(M, M \setminus \{x\}; \mathbb{Z})$. Prove that

(ii) The sets $U(\mu_B)$ provide a basis for a topology of $\tilde{M}$;
(iii) The map p is a two-sheeted covering projection;
(iv) The manifold $\tilde{M}$ is orientable.

Exercise 0.4. Let M be a closed connected non-empty n-manifold.

(i) Show that M is orientable if and only if $\tilde{M}$ has two connected components.
(ii) *Bonus:* Show that any closed connected manifold with simple non-cyclic of order 2 fundamental group is orientable.