Exercise sheet No. 4

Deadline: 11/11/2019.

You should hand only exercises 0.1 and 0.2. Let $\mathcal{R}$ be either $\mathbb{R}$ or $\mathbb{Z}$.

Exercise 0.1 (5 points)**.** Let M and N be two orientable closed connected non-empty manifold of dimension n. Let us consider their fundamental classes $[M] \in H_n(M;\mathbb{Z})$ and $[N] \in H_n(M;\mathbb{Z})$. Let $f\colon M \to N$ be a continuous map such that $H_n(f)([M]) = [N]$. Prove that $H_j(f)$ is surjective for all $j = 0, \cdots, n$ and that $H^j(f)$ is injective for all $j = 0, \cdots, n$.

Exercise 0.2 (5 points)**.** Let $\mathcal{R}$ be a normed ring with respect with the absolute-value norm. Let V be an R-module. We define a norm on V to be a map

$$\|\cdot\|\colon V \to \mathbb{R}$$

such that

(i) $\|v + w\| \leq \|v\| + \|w\|$, for every $v, w \in V$;
(ii) $\|v\| = 0$ if and only if $v = 0$, for every $v \in V$;
(iii) $\|\lambda v\| \leq |\lambda|\|v\|$, for every $\lambda \in \mathcal{R}$ and $v \in V$.

Show that

(a) If $\mathcal{R}$ is a field, then item *(iii)* is an equality;
(b) Assume that item *(iii)* is an equality and that $\mathcal{R} = \mathbb{Z}$. Is it always possible to endow a $\mathbb{Z}$-module with a norm satisfying the previous assumptions?

Exercise 0.3. The aim of this exercise is to compute the 0-th and the 1-st bounded cohomology groups of $\mathbb{S}^1$ with $\mathbb{R}$-coefficients.

(i) Compute $H_b^0(\mathbb{S}^1;\mathbb{R})$;
(ii) Prove that the comparison map $c^1_{\mathbb{S}^1}\colon H_b^1(\mathbb{S}^1;\mathbb{R}) \to H^1(\mathbb{S}^1;\mathbb{R})$ is the zero map;
(ii) Let $\varphi \in C_b^1(\mathbb{S}^1;\mathbb{R})$ be such that $\varphi = \partial f$ with $f \in C^0(\mathbb{S}^1;\mathbb{R})$. Prove that, there exists an $\overline{f} \in C_b^0(\mathbb{S}^1;\mathbb{R})$ such that $\varphi = \partial \overline{f}$.

Exercise 0.4. Compute the 0-th and the 1-st bounded cohomology groups of any path-connected topological space. (*Hint: Use Hurewicz Theorem and Exercise* 0.5 *of Exercise sheet No.*2.)