Exercise sheet No. 5

Deadline: 18/11/2019.

You should hand only exercises 0.1 and 0.2. Let $\mathcal{R}$ be either $\mathbb{R}$ or $\mathbb{Z}$.

Exercise 0.1 (4 points)**.** Let $(V, \|\cdot\|)$ be a normed vector space. Let W be a linear subspace of V. Then, we define a function

$$\rho\colon V/W \to \mathbb{R}$$

by

$$\rho([v]) = \rho(v + W) \coloneqq \inf\{\|v + w\| \mid w \in W\} ,$$

where $[v] \in V/W$ and $v \in V$. Show that ρ is a norm if and only if W is closed.

Exercise 0.2 (6 points)**.** Let Γ_1 and Γ_2 be discrete groups. Let $\psi\colon \Gamma_1 \to \Gamma_2$ be a homomorphism. Suppose that V is a normed Γ_2-module over $\mathcal{R}$. Then, show that

(i) One can endow V with the structure of normed Γ_1-module over $\mathcal{R}$ via ψ. We denote this structure by $\psi^{-1}V$ (*Hint: you should define an action of* Γ_1 *on* V *via the homomorphism* ψ);

(ii) Show that ψ induces a well-defined homomorphism in bounded cohomology

$$H_b^\bullet(\psi)\colon H_b^n(\Gamma_2, V) \to H_b^n(\Gamma_1, \psi^{-1}V) .$$

Definition 0.3. Let Y be a path-connected CW-complex and let Γ be a group acting on Y. We say that the action is a *covering space action* if the following holds: For each $y \in Y$, there exists an open neighbourhood U of y such that $\gamma_1(U) \cap \gamma_2(U) \neq \emptyset$ implies $\gamma_1 = \gamma_2$ for all $\gamma_1, \gamma_2 \in \Gamma$.

Exercise 0.4. Prove that if Γ acts on a simply connected CW-complex Y via a covering space action, then Γ is the group of the deck transformation of the universal covering $Y \to Y/\Gamma$.

Exercise 0.5. Let Γ be a group and let $\mathcal{R} = \mathbb{R}$ with the trivial Γ-action. We construct a space $E\Gamma$ associated to Γ as follows: $E\Gamma$ is the Δ-complex whose n-simplices are defined by the $(n+1)$-tuples of ordered elements $(\gamma_0, \cdots, \gamma_n) \in \Gamma^{n+1}$. We attach such a simplex to an $(n-1)$-simplex of the form $(\gamma_0, \cdots, \hat{\gamma_i}, \cdots, \gamma_n)$, where $0 \leq i \leq n$. Show that

(i) the Δ-complex $E\Gamma$ is contractible;

Consider now the cochain complex

$$(C^\bullet(\Gamma, \mathcal{R}) = \{f\colon \Gamma^{\bullet+1} \to \mathcal{R}\}, \delta^\bullet)$$

with the usual boundary operator. Show that

(ii) the cohomology $H^k((C^\bullet(\Gamma, \mathcal{R}), \delta^\bullet))$ is isomorphic to the *simplicial* cohomology of $E\Gamma$ (i.e. the cochains are defined on the simplices of $E\Gamma$ instead on singular simplices).

Conclude that

(iii) If we don't restrict to the Γ-invariants of $C^\bullet(\Gamma, \mathcal{R})$, when we consider cohomology of groups, we get a trivial cohomology.

Now notice that

(iv) Γ acts on $E\Gamma$ freely and simplicially (i.e. it sends each simplex onto another simplex via linear homeomorphism).

Show that this implies that

(v) Γ acts on $E\Gamma$ via a covering space action.

The previous condition easily implies that the quotient $B\Gamma$ is still a Δ-complex. Using Exercise 0.4, show that

(vi) The space $B\Gamma$ has the fundamental group isomorphic to Γ and the higher homotopy groups of $B\Gamma$ are trivial (i.e. $B\Gamma$ is a $K(\Gamma, 1)$-space).

Using this construction, prove that

(vii) The simplicial cohomology of $B\Gamma$ with $\mathcal{R}$-coefficients is isomorphic to $H^\bullet(\Gamma, \mathcal{R})$.

Since simplicial cohomology is isomorphic to singular cohomology we have proved that

$$H^\bullet(B\Gamma; \mathcal{R}) \cong H^\bullet(\Gamma, \mathcal{R}) \ .$$

However, in general, the previous construction does not carry over to the bounded context.

(viii) *Bonus*: Provide an example of simplicial complex in which the simplicial bounded cohomology is different to the singular bounded cohomology.