Exercise sheet No. 6

Deadline: 25/11/2019.

You should hand only exercises 0.5 and 0.6.

We recall here some theorems about homotopy groups that are needed in the sequel.

Theorem 0.1 (Freudenthal suspesion theorem)**.** *The suspension map* $\pi_i(\mathbb{S}^n) \to \pi_{i+1}(\mathbb{S}^{n+1})$ *is an isomorphism for* $i < 2n-1$ *and a surjection for* $i = 2n-1$.

Definition 0.2. A map between CW-complex $f\colon X \to Y$ is said to be *cellular* if $f(X^n) \subset Y^n$, where X^n and Y^n denote the n-skeletons of X and Y, respectively.

Theorem 0.3 (Cellular approximation theorem)**.** *Every map between CW-complexes is homotopic to a cellular map.*

Theorem 0.4. *If we have a fibration* $F \hookrightarrow E \xrightarrow{p} B$, *then it induces a long exact sequences in homotopy:*

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \cdots ,$$

where b_0 *is a base point of* B *and* $x_0 \in p^{-1}(b_0)$.

Exercise 0.5 (5 points)**.** Show that

(i) The Klein bottle is a $K(G,1)$-space, where $G = \langle a, b | aba = b\rangle$.

(ii) Are there other surfaces which are $K(G,n)$ for some G and $n \geq 1$?

(iii) Are there $K(G,1)$-spaces which are closed n-manifolds with $n \geq 3$?

Exercise 0.6 (5 points)**.** Show that

(i) $\pi_n(\mathbb{S}^k) = 0$ if $n < k$.

Deduce that

(ii) $\pi_n(\mathbb{S}^n) \cong \mathbb{Z}$;

(iii) $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$.

Exercise 0.7. Show that the inhomogeneous chain complex (i.e. the one associated to the bar resolution) is chain isomorphic to the homogeneous one.

Exercise 0.8. Let X be a CW-complex and let X' be a CW-complex obtained from X by attaching a cell $f\colon \mathbb{D}^n \to X$ along the map $f|_{\mathbb{S}^{n-1}}\colon \mathbb{S}^{n-1} \to X$. Let $\alpha = [f|_{\mathbb{S}^{n-1}}] \in \pi_{n-1}(X)$. Show that

(i) $\pi_{n-1}(i)(\alpha) = 0 \in \pi_{n-1}(X')$, where $i\colon X \to X'$ is the inclusion.

Describe

(ii) how to construct for all $n \geq 1$ a CW-complex $K(\mathbb{Z}, n)$ starting from the sphere $\mathbb{S}^n$.

Show that if X is an n-dimensional CW-complex, then

(iii) The set of homotopy classes $[X, \mathbb{S}^n]$ is in bijection with the one $[X, K(\mathbb{Z}, n)]$.

Let $G\langle X|R\rangle$ be a presentation of the group G, where X denotes the set of generators and R the set of relations. Show that

(iv) If G is finitely generated, then there exists a CW-complex $K(G, 1)$ whose 1-skeleton is finite;

(v) If G is finitely presented, then there exists a CW-complex $K(G, 1)$ whose 2-skeleton is finite.