Exercise sheet No. 7

Deadline: 02/12/2019.

You should hand only exercises 0.3 and 0.4.

Definition 0.1. Let $F_2 = \langle a, b\rangle$ be the free non-Abelian group of rank 2 and let

$$\ell^\infty_{\text{odd}}(\mathbb{Z}) \coloneqq \{\alpha\colon \mathbb{Z} \to \mathbb{R} \mid \|\alpha\|_\infty < +\infty, \alpha(n) = -\alpha(-n) \text{ for all } n \in \mathbb{Z}\} \ .$$

For every $\alpha \in \ell^\infty_{\text{odd}}(\mathbb{Z})$ we consider the map

$$q_\alpha\colon F_2 \to \mathbb{R}$$

defined by

$$f_\alpha(a^{n_1}b^{m_1}\cdots a^{n_k}b^{m_k}) = \sum_{i=1}^{k} f(n_i) + f(m_i) \ ,$$

where we are identifying each element of F_2 with the unique word in reduced form representing it (we allow n_1 and m_k to be 0).

The map q_α is called *Rolli's quasi-morphism.*

Definition 0.2. Let $F_2 = \langle a, b\rangle$ be the free non-Abelian group of rank 2 and let $\omega \in F_2$ be a reduced word. We define

$$q_\omega\colon F_2 \to \mathbb{R}$$

by

$$q_\omega(g) = \#\text{ occurencies of } \omega \text{ in } g - \#\text{ occurencies of } \omega^{-1} \text{ in } g \ ,$$

where g is written in reduced form (e.g. if $\omega = abab$, then $q_\omega(ababab) = 2 - 0 = 2$).

The map q_ω is called *Brooks quasi-morphism.*

Exercise 0.3 (4 points)**.** Show that

(i) For every $\alpha \in \ell^\infty_{\text{odd}}(\mathbb{Z})$, q_α is a quasi-morphism and compute its defect.

(ii) For every reduced word $\omega \in F_2$, q_ω is a quasi-morphism and compute its defect.

Exercise 0.4 (6 points)**.** We consider the definition of *quasihomomorphism* introduced by Fujiwara and Kapovich in "On quasihomomorphism with noncommutative targets" given at line 4 (you may find it on arXiv: https://arxiv.org). Let G and H be discrete groups. A map $f\colon G \to H$ is called *quasihomomorphism* if the set of *defects* of f

$$D(f) = \{f(y)^{-1}f(x)^{-1}f(xy) \mid x, y \in G\}$$

is finite. Show that

(i) If the target is $\mathbb{Z}$, we obtain the usual notion of quasi-morphism.

Read Definition 2.1 of the paper above.

(ii) Why is any almost homomorphism a quasihomomorphism?

Show that

(iii) The composition of quasihomomorphisms is still a quasihomomorphism;

Consider now the composition $f_1 \circ f_2$, where f_2 is a quasihomomorphism and f_1 is a homomorphism.

(iv) Which additional condition should f_1 satisfy in order to show that $f_1 \circ f_2$ is a homomorphism?

Show that

(v) If G is finitely generated and $f\colon G \to H$ is a quasihomomorphism, then also the group generated by the image $\langle f(G) \rangle$ is finitely generated.

Exercise 0.5. Prove that

(i) The space of quasi-morphism $Q(\Gamma, \mathbb{R})$ is a vector space over $\mathbb{R}$ and that homogeneous quasi-morphisms are a subspace.

Show that

(ii) The quotient $Q(\Gamma, \mathbb{R})/\mathrm{Hom}(\Gamma, \mathbb{R})$ is a normed vector space, where the norm is given by the defect.

Let us consider $Q(\mathbb{Z}, \mathbb{Z})$. Show that

(iii) $Q^h(\mathbb{Z}, \mathbb{Z})$ is not dense in $Q(\mathbb{Z}, \mathbb{Z})$ with respect to the ℓ^∞-norm.

Definition 0.6. A *central extension* of Γ by $\mathcal{R}$ (here $\mathcal{R}$ is either $\mathbb{Z}$ or $\mathbb{R}$ with the trivial action) is an exact sequence

$$1 \to \mathcal{R} \xrightarrow{\iota} \Gamma' \xrightarrow{\pi} \Gamma \to 1$$

such that $\iota(\mathcal{R})$ is contained in the center of Γ'.

Exercise 0.7. Given a set theoretic section $s\colon \Gamma \to \Gamma'$ of π, show that

(i) There exists a well-defined map

$$\varphi\colon \Gamma \times \Gamma \to \mathcal{R}$$
$$\varphi(g_1, g_2) = s(g_1 g_2)^{-1} s(g_1) s(g_2) \ .$$

Show that

(ii) The map $\Gamma' \to \mathcal{R} \times \Gamma$ given by $g \mapsto (g \cdot s(\pi(g))^{-1}, \pi(g))$ is a bijection;

(iii) Using the previous map show that the group law in Γ' translates to

$$(\mathcal{R} \times \Gamma) \times (\mathcal{R} \times \Gamma) \to \mathcal{R} \times \Gamma$$
$$((a, g), (b, h)) \mapsto (a + b + \varphi(g, h), gh) \ .$$

Using the associativity of the previous group law, prove that

(iv) $\varphi \in \overline{Z}^2(\Gamma; \mathcal{R})$.

Show that

(v) The previous construction does not depend on the chosen section, i.e. different sections provide cohomologous cocycles.

Conclude that

(vi) Any central extension $\mathcal{C}$ of Γ by $\mathcal{R}$ defines an element $e(\mathcal{C}) \in H^2(\Gamma; \mathcal{R})$.