Exercise sheet No. 8

Deadline: 09/12/2019.
You should hand only exercises 0.1 and 0.2.

Exercise 0.1 (5 points)**.** Prove that $F_2 = \langle a, b \rangle$ is not amenable (using the definition of amenable groups via left-invariant means).

Exercise 0.2 (5 points)**.** Let Σ_g be a surface of genus $g \geq 2$. Prove that

$$H_b^2(\pi_1(\Sigma_g); \mathbb{R})$$

is infinite dimensional. Here $\mathbb{R}$ is endowed with the trivial action.

Exercise 0.3. Consider $\ell^\infty(\Gamma)$ with the following Γ-action:

$$\Gamma \times \ell^\infty(\Gamma) \to \ell^\infty(\Gamma)$$
$$(\gamma, f) \mapsto \gamma \cdot f(g) := f(\gamma^{-1} g) \ ,$$

where $g \in \Gamma$. Show that

(i) $\ell^\infty(\Gamma)$ is a Banach Γ-module over $\mathbb{R}$.

Moreover, prove that

(ii) If Γ admits a left-invariant mean, then it also admits a bi-invariant mean, i.e. a mean m such that

$$m(f \cdot \gamma) = m(f) \text{ and } m(\gamma \cdot f) = m(f) \ ,$$

for every $f \in \ell^\infty(\Gamma)$ and $\gamma \in \Gamma$ (here, $f \cdot \gamma(g) = f(g\gamma^{-1})$, for every $g \in \Gamma$).

Exercise 0.4. Prove that

(i) A chain cocomplex $(C^k, \delta^k)_{k \geq 0}$ of free modules over $\mathbb{R}$ is acyclic (i.e. $\mathrm{Im}(\delta^k) = \ker(\delta^{k+1})$) if and only if there exists a family of maps $h_k \colon C^k \to C^{k-1}$ such that

$$\delta^{k-1} \circ h_k + h_{k+1} \circ \delta^k = \mathrm{Id}_{C^k}.$$

Let Γ be a discrete group and let V be an Γ-module over $\mathbb{R}$. Show that

(ii) The complex $(C^\bullet(\Gamma, V))$ is acyclic by constructing a family of maps as above (we have already seen several proofs of this fact!).