Exercise sheet No. 9

Deadline: 16/12/2019.

You should hand only exercises 0.3 and 0.4.

Recall the following definitions that will be useful in the sequel.

Definition 0.1. A group Γ is *virtually solvable* if it contains a solvable subgroup of finite-index .

Definition 0.2. A group Γ is *residually finite* if for each element $g \neq e \in \Gamma$ there exists a finite-index normal subgroup $N(g) \trianglelefteq \Gamma$ which does not contain g.

Exercise 0.3 (5 points)**.** Given the following short exact sequence

$$1 \to K \to \Gamma \to H \to 1 \ ,$$

prove that

(i) If H and K are amenable, then also Γ is so.

(ii) Prove that virtually solvable groups are amenable.

Exercise 0.4 (5 points)**.** Prove that

(i) If Γ is amenable and $H \leq \Gamma$, then H is amenable.

Show that

(ii) A group is a subgroup of an infinite direct product of non-trivial finite groups if and only if it is residually finite (*Hint: it could be useful to express the notion of residually finite groups in terms of certain homomorphisms onto finite groups*).

(iii) *Bonus:* Free groups are residually finite.

Conclude (you may use item (*iii*)) that

(iv) In general, the infinite direct product of amenable groups is not amenable.

Definition 0.5. A discrete countable group Γ satisfies the *Følner condition* if for every finite subset $A \subset \Gamma$ and every $\varepsilon > 0$ there exists a finite non-empty subset $F \subset \Gamma$ such that for each $a \in A$ we have

$$\frac{|aF \bigtriangleup F|}{|F|} \leq \varepsilon \ ,$$

where $\bigtriangleup$ denotes the symmetric difference of sets.

Given a discrete and countable group Γ, a *Følner sequence* is a sequence $\{F_n\}_{n \in \mathbb{N}}$ of non-empty and finite subsets of Γ such that

$$\frac{|gF_n \bigtriangleup F_n|}{|F_n|} \to 0 \ ,$$

for every $g \in \Gamma$.

Exercise 0.6. Prove that

(i) A discrete countable group Γ satisfies the Følner condition if and only if it has a Følner sequence.

(ii) $\mathbb{Z}$ has a Følner sequence;

(iii) Every finitely generated Abelian group has a Følner sequence;
(iv) If every finitely generated subgroup of a discrete countable group Γ satisfies the Følner condition, then Γ does so;
(v) All Abelian groups satisfy the Følner condition.
(vi) *Bonus:* Provide some intuitive explanations that the free groups of rank at least 2 does not have any Følner sequence.

Remark 0.7. One can prove that a group Γ is amenable if and only if it satisfies the Følner condition (or, equivalently, it has a Følner sequence). So the previous exercise gives a proof of the mentioned fact that Abelian groups are amenable. Moreover, we provided a new proof of the fact that free groups of rank at least 2 are not amenable.