# Higher Invariants Oberseminar (HIOB)
## ***Aspherical Manifolds***

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### Time and Place

The **first introductory meeting** on Monday, 18.10.2021, will be online via Zoom starting at 12.30. We will decide in the first meeting whether HIOB will continue online via Zoom or in-person in the SFB Seminar Room (possibly in hybrid format).

Zoom Details (for the first meeting):
*Zoom Meeting ID*: 651 6501 1479
*Passcode*: 342677

After the first meeting, HIOB will continue as usual every Monday at 12.15–13.45.

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### *Schedule & Overview of Talks*

*Preliminary Reading*: We recommend browsing the lecture notes by M. W. Davis [Dav01] and F. T. Farrell [Far02] and the survey by W. Lück [Lüc10].

#### *Introduction*

**Talk 1** (18.10.2021): *Introduction and Overview of the Talks.*

**Talk 2** (25.10.2021): *First examples and properties of aspherical manifolds.* The goal of this talk is to discuss classical examples of aspherical manifolds and recall some basic properties of aspherical spaces/manifolds. Most of the material for this talk is classical and can be found in many places in the literature (some of this material might also be covered in Talk 1).

Briefly review the existence and homotopical uniqueness of aspherical spaces for a given fundamental group and discuss the homotopy classification of maps between aspherical spaces. Prove that the fundamental group of an aspherical finite-dimensional CW-complex is torsionfree [Lüc10, Lemma 3.1]. Explain under which assumptions the glueing of two aspherical spaces produces again an aspherical space [Whi39, Corollary, p. 160].

Show that Riemannian manifolds with non-positive sectional curvature are examples of aspherical manifolds using the Cartan–Hadamard Theorem (see, e.g., [Lee18, Theorem 12.8, p. 352]). Identify the closed connected aspherical manifolds in dimensions 1 and 2, and discuss briefly asphericity for closed 3-manifolds [Lüc10, Section 2.2]. Time permitting, one could also mention further examples, for example, those arising from quotients of torsionfree lattices in almost connected Lie groups [Lüc10, Section 2.3]; see also [Dav14, Section 1].

(Optional) Show that the connected sum of two closed connected aspherical manifolds, which are not homotopy equivalent to a sphere, is never aspherical in dimension $n \geq 3$ [Lüc10, Lemma 3.2].

*Davis' reflection group trick and its applications*

The goal of this section is to discuss Davis' reflection trick. This is a useful construction which produces aspherical manifolds with *exotic* properties. The first version of the trick appeared in Davis' paper [Dav83] but other variations and alternative approaches are also available in the literature, see, for example, [Dav01]. We will follow the classical version of the *reflection group trick* as presented in Davis' book [Dav12]. However, we also recommend the previous two references as useful complementary sources.

**Talk 3** (08.11.2021): *Coxeter groups.* The goal of this talk is to recall the basic definitions and properties of Coxeter groups that we will need in the sequel. Begin with the definition of Coxeter groups following [Dav12, Section 3.3]. Explain the equivalent definitions as in [Dav12, Theorem 3.3.4] (the proof can be omitted). An important source of examples are dihedral groups which are discussed in [Dav12, Section 3.1]. Discuss the main combinatorial lemmas that will be needed in the sequel [Dav12, Lemmas 4.7.2 and 4.7.3]. Present some further examples from [Dav12, Chapter 6, esp. Section 6.5]. Discuss the canonical representation [Dav12, Section 6.12].

**Talk 4** (15.11.2021): *The space* $\mathcal{U}(G, X)$. In this talk we will discuss the construction of $\mathcal{U}(G, X)$ and its properties. Begin by defining the space $\mathcal{U}(G, X)$ associated to a Coxeter group $G = (W, S)$ and a mirrored space $X$ over $S$ [Dav12, Section 5.1]. Then, discuss the nerve of a Coxeter system following [Dav12, Section 7.1] and prove [Dav12, Theorem 7.2.4]. Pick and briefly discuss some examples of this construction from [Dav12, Section 7.4]. Finally, sketch the proof of [Dav12, Theorem 9.1.4] which combines [Dav12, Corollary 8.2.2 and Theorem 9.1.3]. (Also, explain how [Dav12, Lemma 4.7.3] from the previous talk is needed for the proof of [Dav12, Lemma 8.1.3].)

**Talk 5** (22.11.2021): *Exotic aspherical manifolds.* In this talk we will construct examples of closed aspherical manifolds which are not covered by Euclidean spaces [Dav12, Theorem 10.5.1]. The proof builds on the notion of the fundamental group at infinity [Dav12, Chapter 9; Section 9.2, esp. Proposition 9.2.2 and Example 9.2.7] and certain properties of homology spheres [Dav12, Sections 10.3–10.4]. After reviewing these results, discuss the proof of [Dav12, Theorem 10.5.1] and explain [Dav12, Remark 10.5.2]. See also [Dav01, Section 9].

**Talk 6** (29.11.2021): *Applications of the reflection group trick.* Discuss the (first version of the) reflection group trick [Dav12, Section 11.1] and explain the construction of aspherical manifolds with "exotic" fundamental groups [Dav12, Section 11.2]. Then, after reviewing the necessary prerequisites from smoothing theory, explain the construction of nonsmoothable aspherical manifolds [Dav12, Section 11.3]. See also [Dav01, Sections 12–13].

*Hyperbolization of polyhedra*

This part is concerned with the hyperbolization constructions of Gromov and Davis–Januszkiewicz. We will mostly follow the paper of Davis–Januskiewicz [DJ91]. Some useful complementary sources are Davis' survey [Dav01, Sections 14–17] and the paper of Edmonds–Klee [EK15, Section 3].

**Talk 7** (06.12.2021): *The Williams functor.* First give a quick overview of the goal of [DJ91] following [DJ91, Introduction]. Then, the talk will be devoted to the study of the Williams functor [DJ91, Section 1]. Describe carefully the construction in [DJ91, Sections 1.a–1.d]. Then explain the local structure of the hyperbolization, as well as its tangential properties, following [DJ91, Sections 1.e–1.f]. Finally, explain the construction in [DJ91, Section 1.h]. If time permits, discuss also the relative situation [DJ91, Section 1.g].

**Talk 8** (13.12.2021): *Hyperbolization of simplices.* Introduce the notion of hyperbolized simplices and prove [DJ91, Theorem 4a.3]. The remaining part of the talk will be devoted to the proof that hyperbolized simplices actually exist. Constructions are discussed in detail in [DJ91, Sections 4b and 4c]. Then deduce [DJ91, Introduction, Theorems A and B]. See also [Dav01, Sections 14–15].

**Talk 9** (20.12.2021): *Applications of hyperbolization.* In this talk we will construct exotic examples of aspherical manifolds via the hyperbolization of polyhedra. The first construction [DJ91, Section 5a] gives an example of a non-triangulable aspherical manifold. Before proving [DJ91, Theorem 5a.1], first explain the construction of $M(E_8)$ [Bro72, pp. 116-126] and review (as thoroughly as time permits) the required results used in the proof (e.g. Rohlin's Theorem); see also [Dav01, Section 16]. Finally, deduce [DJ91, Corollary 5a.4]. If time permits, discuss also the construction of nonpositively curved topological manifolds which are not covered by Euclidean spaces following [DJ91, Section 5b].

*PD–groups and the Borel conjecture*

**Talk 10** (10.01.2022): *Poincaré duality groups.* Introduce the notion of a (virtual) Poincaré duality group [Dav12, Section 10.9] and discuss some of their properties [Dav14, Sections 2–4]. Choose and present some examples from [Dav14, Sections 5 and 7]. Sketch the proof that there are Poincaré duality groups which are not be finitely presented in every dimension $n \geq 4$ [Dav14, Theorem 7.15]. The proof can be found in [Dav12, Theorem 11.6.4 and Corollary 11.6.5] using the (second version of the) reflection group trick [Dav12, Section 11.5, esp. Proposition 11.5.3] and the construction of the Bestvina–Brady groups [Dav12, Section 11.6].

**Talk 11** (17.01.2022): *The Borel conjecture.* State and discuss the Borel conjecture following [KL05, Chapter 1], [Dav12, Appendix H], [Far02]. Survey the approach to the conjecture via algebraic $K$-theory and $L$-theory and explain the connection with the Novikov conjecture [KL05], [Dav12, Appendix H]. Outline some of the ideas of the proof in the case of non-positively curved Riemannian manifolds [Far02, Sections 2–4].

**Talk 12** (24.01.2022): *The Hopf conjecture and the work of Charney–Davis.* The aim of the talk is to discuss the Hopf conjecture on the sign of the Euler characteristic of non-positively curved Riemannian manifolds (and its variations). The talk will mainly be based on the paper by Charney–Davis [CD95]. Explain the material in [CD95, Sections 2–3], reviewing all the necessary background, and proceed to discuss the conjectures of Charney–Davis [CD95, Sections 4–6] and their relation to Hopf's conjecture [CD95, Proposition 6.5]. If time permits, the (connection with the) Singer conjecture should also be mentioned (see, e.g., [Lüc10, Section 9], [Dav08]).

**Talk 13** (31.01.2022): *The Chern conjecture.* The Chern conjecture asserts that every closed affine manifold has zero Euler characteristic. The goal of this talk is to discuss this conjecture in the case of closed affine *aspherical* manifolds following [Fri17, Chapters 12–13, esp. 13.4], [Kli17], [BCL18]. Follow the discussion in [Fri17, Section 13.4], [Kli17, Section 1] ([Fri17] also contains all the necessary background about the simplicial volume) and then present the main result of Bucher–Connell–Lafont [BCL18] and its relation to Chern's conjecture [BCL18, Remark 1.4 and Corollary 1.5]. The speaker may decide whether to devote the talk to the proof of [BCL18, Main Theorem] or to an outline of Klingler's proof of the Chern conjecture in the case of special affine manifolds [Kli17].

**Talk 14** (07.02.2022): *Discussion for the next HIOB.* (Alternatively, we may continue with another talk related to either Talk 11 or Talk 13 and postpone the *Discussion* until the following week.)

## References

[BCL18] M. Bucher, C. Connell, and J. F. Lafont, *Vanishing simplicial volume for certain affine manifolds*, Proc. Amer. Math. Soc. **146** (2018), 1287–1294.

[Bro72] W. Browder, *Surgery on simply connected manifolds*, Springer, Berlin, 1972.

[CD95] Ruth Charney and Michael Davis, *The Euler characteristic of a nonpositively curved, piecewise Euclidean manifold*, Pacific Journal of Mathematics **171** (1995), no. 1, 117 – 137.

[Dav83] Michael W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. **117** (1983), no. 2, 293–324.

[Dav01] M. W. Davis, *Exotic aspherical manifolds*, `http://users.ictp.it/~pub_off/lectures/lns009/Davis/Davis_new.pdf`, 2001.

[Dav08] ———, *The Hopf Conjecture and the Singer Conjecture*, Guido's Book of Conjectures (I. Chatterji, ed.), Monographie de L'Enseignement Math., vol. 40, Princeton University Press, 2008, pp. 80–82.

[Dav12] Michael W. Davis, *The Geometry and Topology of Coxeter Groups. (LMS-32)*, Princeton University Press, 2012.

[Dav14] M. W. Davis, *Poincaré duality groups*, Surveys on Surgery Theory (AM-145), vol. 1, Princeton University Press, 2014, pp. 167–194.

[DJ91] Michael W. Davis and Tadeusz Januszkiewicz, *Hyperbolization of polyhedra*, J. Differential Geom. **34** (1991), no. 2, 347–388.

[EK15] A. L. Edmonds and S. Klee, *The combinatorics of hyperbolized manifolds*, Mathematica Scandinavica **117** (2015), no. 1, 31–63.

[Far02] F. T. Farrell, *The Borel conjecture*, Topology of high-dimensional manifolds. Proceedings of the school on high-dimensional manifold topology, Abdus Salam ICTP, Trieste, Italy, May 21–June 8, 2001. Number 1 and 2, Trieste: The Abdus Salam International Centre for Theoretical Physics, 2002, pp. 225–298.

[Fri17] Roberto Frigerio, *Bounded cohomology of discrete groups*, vol. 227, American Mathematical Soc., 2017.

[KL05] Matthias Kreck and Wolfgang Lück, *The Novikov conjecture. Geometry and algebra*, vol. 33, Basel: Birkhäuser, 2005 (English).

[Kli17] Bruno Klingler, *Chern's conjecture for special affine manifolds*, Ann. Math. (2) **186** (2017), no. 1, 69–95.

[Lee18] John Lee, *Introduction to Riemannian Manifolds*, Graduate Text in Mathematics, vol. 176, Springer International Publishing, 2018.

[Lüc10] Wolfgang Lück, *Survey on aspherical manifolds*, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2010, pp. 53–82.

[Whi39] J. Whitehead, *On the asphericity of regions in a 3-sphere*, Fund. Math. **32** (1939), no. 1, 149–166.