Gromov's systolic inequality via smoothing technique

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Outline

1. Gromov's systolic inequality and simplicial volume
2. Smoothing technique and Gromov's smoothing inequality

Main references:

- F. Balacheff and S. Karam, Macroscopic Schoen conjecture for manifolds with nonzero simplicial volume, Trans. Amer. Math. Soc. **372** (2019), no. 10, 7071–7086.
- M. Gromov, Volume and bounded cohomology, Inst. Hautes Études Sci. Publ. Math. No. 56 (1982), 5–99 (1983).

Let M be a closed n-dimensional Riemannian manifold endowed with a Riemannian metric $\mathcal{G}$, denoted $(M, \mathcal{G})$.

Definition

The systole of a Riemannian manifold $(M, \mathcal{G})$, denoted by $\operatorname{Sys}\pi_1(M, \mathcal{G})$, is defined to be the shortest length of a non-contractible loop.

Definition

A closed n-dimensional manifold M is called essential, if there exists a continuous map from M to an aspherical space K, so that $f_*([M]) \neq 0$ in $H_n(K; R)$.

Example of essential manifolds:

- Aspherical manifolds: hyperbolic n-manifolds, n-torus $\mathbb{T}^n$
- Real projective n-spaces $\mathbb{RP}^n$
- Connected sums: $\mathbb{T}^3 \# (S^2 \times S^1)$

Theorem (Gromov 1983, [6])

Let M be a closed n-dimensional essential manifold. Then any Riemannian metric $\mathcal{G}$ defined on M satisfies

$$\operatorname{Sys} \pi_1(M, \mathcal{G})^n \leqslant C_n \operatorname{Vol}_{\mathcal{G}}(M), \tag{1}$$

where C_n is a constant only depending on the dimension n.

Let $\|M\|$ be simplicial volume of a closed manifold M.

Theorem (Gromov 1983, see [6])

For a closed essential n-dimensional manifold M, the optimal constant in systolic inequality (1) is related to simplicial volume as follows,

$$\operatorname{Sys}\pi_1(M,\mathcal{G})^n \leqslant D_n \frac{\log^n(1+\|M\|)}{\|M\|} \operatorname{Vol}_{\mathcal{G}}(M),$$

where D_n is a constant only depending on n.

Definition

The systolic volume of a closed n-dimensional manifold M, denoted $\mathrm{SR}(M)$, is defined to be

$$\inf_{\mathcal{G}} \frac{\mathrm{Vol}_{\mathcal{G}}(M)}{\mathrm{Sys}\,\pi_1(M,\mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on M.

Theorem (Babenko 1992)

1. *Let M be a closed orientable n-manifold. If $\mathrm{SR}(M) > 0$, then M is essential.*
2. *Systolic volume is a homotopy invariant.*

Systolic volume and other topological invariants

Let M be a closed n-dimensional essential manifold.

1. **M. Brunnbauer 2008**, Minimal volume entropy $\lambda(M)$:

$$\mathrm{SR}(M) \geqslant D_n \frac{\lambda(M)}{\log^n(1+\lambda(M))},$$

   where D_n is a constant only depending on n.

2. **M. Gromov 1983**, Betti numbers $b_k(M;\mathbb{F})$:

$$\mathrm{SR}(M) \geqslant E_n \frac{b_k(M;\mathbb{F})}{\exp\left(E'_n\sqrt{\log b_k(M;\mathbb{F})}\right)},$$

   where E_n and E'_n are two constants only depending on n.

Let $(M, \mathcal{G})$ be a Riemannian manifold. The filling radius, injectivity radius, convex radius are denoted by $\mathrm{FillRad}(M, \mathcal{G})$, $\mathrm{Inj}(M, \mathcal{G})$, $Conv(M, \mathcal{G})$.

Proposition

There holds

$$6\, \mathit{FillRad}(M, \mathcal{G}) \geqslant \mathrm{Sys}\, \pi_1(M, \mathcal{G}) \geqslant 2\, \mathrm{Inj}(M, \mathcal{G}) \geqslant 4\, \mathit{Conv}(M, \mathcal{G}).$$

Let $I(M, \mathcal{G})$ be one of the invariants: $\mathrm{FillRad}(M, \mathcal{G})$, $\mathrm{Sys}\, \pi_1(M, \mathcal{G})$, $\mathrm{Inj}(M, \mathcal{G})$, $Conv(M, \mathcal{G})$. Define

$$\mathrm{IV}(M) = \inf_{\mathcal{G}} \frac{\mathrm{Vol}_{\mathcal{G}}(M)}{I(M, \mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on M.

Problem: What is the relation between the constant $IV(M)$ and topology of the manifold M?

The constant $\mathrm{IV}(M)$ **may represent topological complexity of the manifold.**

Some known results:

- **M. Brunnbauer 2008** For $\mathrm{FillRad}(M, \mathcal{G})$: the constant $\mathrm{IV}(M)$ does not depend on topology of the manifold.

- For $\mathrm{Inj}(M, \mathcal{G})$,
  - **Yamaguchi 1988** For any positive constant C, in the set

$$\{\mathrm{IV}(M) \leqslant C | M \text{ is a compact } n\text{-dimensional manifold}\},$$

    there are only finitely many homotopy types.
  - **Chen 2019** For the Betti number $b_k(M; \mathbb{F})$,

$$\mathrm{IV}(M) \geqslant L_n \frac{b_k(M; \mathbb{F})}{\log\left(L'_n \sqrt{\log b_k(M; \mathbb{F})}\right)},$$

    where M is a compact n-manifold, and L_n and L'_n are two constants only depending on n.

Smoothing technique and Gromov's smoothing inequality

Gromov's inequality

$$\operatorname{Sys}\pi_1(M,\mathcal{G})^n \leqslant D_n \frac{\log^n(1+\|M\|)}{\|M\|}\operatorname{Vol}_{\mathcal{G}}(M)$$

for closed essential n-manifolds M with nonzeo simplicial volume is proved by using "smoothing technique".

Let $(M,\mathcal{G})$ be a Riemannian manifold, $(\widetilde{M},\widetilde{\mathcal{G}})$ be the Riemannian universal covering.

Denote by $\mathcal{M}$ the Banach space of finite measures on $\widetilde{M}$, and by $\mathcal{P}\subset\mathcal{M}$ the subset of probability measures.

A smoothing operator is a smooth map $\mathscr{S}:\widetilde{M}\to\mathcal{P}$.

Theorem (Gromov's smoothing inequality)

Let $(M, \mathcal{G})$ be an n-dimensional Riemannian manifold. The simplicial volume satisfies

$$\|M\| \leqslant n! \|d\mathscr{S}\|_\infty^n \operatorname{Vol}_{\mathcal{G}}(M).$$

Simplicial volume of manifolds

Let M be a closed n-dimensional manifold, $[M] \in H_n(M;\mathbb{R})$ be the fundamental class of real coefficient.

Definition

The simplicial volume of M, denoted $\|M\|$, is defined to be

$$\inf \left\{ \sum_{i=1}^{k} |\lambda_i| \,\middle|\, \sum_{i=1}^{k} \lambda_i \sigma_i \text{ is a cycle representing } [M] \right\},$$

where the infimum is taken over all cycles $\sum_{i=1}^{k} \lambda_i \sigma_i$ representing $[M]$.

Notation: denote by $\mathcal{V}_n$ the maximal volume of an ideal n-simplex in hyperbolic space $\mathbb{H}^n$.

Theorem

If M is a closed hyperbolic n-manifold, then

$$\|M\| = \frac{\mathrm{Vol}_{\mathrm{hyp}}(M)}{\mathcal{V}_n}.$$

Dual principle

For a cohomological class $\Omega \in H^n(M;\mathbb{R})$, set

$$\|\Omega\|_\infty = \inf_{\omega} \sup_{\sigma} \omega(\sigma),$$

where the supremum is taken over all n-simplices σ, and the infimum is taken over all cocycles ω representing Ω.

Proposition (Gromov)

$$\|M\| = \sup\left\{ \Omega([M]) \,|\, \Omega \in H^n(M;\mathbb{R}) \textit{ and satisfying } \|\Omega\|_\infty = 1 \right\}.$$

In particular, if Ω_M is the dual fundamental class of M,

$$\|M\| = \frac{1}{\|\Omega_M\|_\infty}.$$

Alternative definition of simplicial volume

Let (M, hyp) be a hyperbolic manifold (hyp is the hyperbolic metric defined on M). Define a cocycle ω_{hyp} on M as follows,

$$\omega_{\mathsf{hyp}}(\sigma) = \frac{\mathrm{Vol}_{\widetilde{\mathsf{hyp}}}((\tilde{\sigma})_{st})}{\mathrm{Vol}_{\mathsf{hyp}}(M)},$$

where $\tilde{\sigma}$ is any lift of the n-simple σ.

Notation: let $\pi : (\mathbb{H}^n, \widetilde{\text{hyp}}) \to (M, \text{hyp})$ be the Riemannian universal covering.

Proposition

The cocycle $\pi^*\omega_{\text{hyp}}$ is straight,

$$\pi^*\omega_{\text{hyp}}(\widetilde{\sigma}) = \pi^*\omega_{\text{hyp}}(\widetilde{\sigma}_{st}).$$

Hence $\pi^*\omega(\widetilde{\sigma})$ only depends on $n+1$ vertices of $\widetilde{\sigma}$, and the induced function

$$\pi^*\omega_{\text{hyp}} : \mathbb{H}^{n+1} \to \mathbb{R}$$

is continuous and Borel.

Let $(M, \mathcal{G})$ be an n-dimensional Riemannian manifold, $\pi : \widetilde{M} \to M$ be Riemannian universal covering.

Definition (Straight invariant fundamental cocycle)

The straight invariant fundamental cocycle $\tilde{\omega}$ is a cochain of $C(\widetilde{M}, \mathbb{R})$, and satisfies

- Invariance: $\pi_1(M)$-invariant;
- Fundamental cocycle: the only cochain ω satisfying $\pi^*(\omega) = \tilde{\omega}$ is the one representing dual fundamental class Ω_M.
- Straight and Borel: $\tilde{\omega}$ is straight, and the induced function on $\widetilde{M}^{n+1}$ is Borel.

The alternative definition of simplicial volume is

$$\|M\|' = \frac{1}{\inf \|\tilde{\omega}\|_\infty},$$

where the infimum is taken over all straight invariant fundamental cocyles $\tilde{\omega}$.

Theorem

If (M, hyp) *is a hyperbolic manifold, then*

$$\|M\|' = \frac{\text{Vol}_{\text{hyp}}(M)}{\mathcal{V}_n}.$$

Let $\mathcal{M}$ be the space of all finite measures on $\widetilde{M}$, and $\mathcal{P} \subset \mathcal{M}$ be the subspace of all probability measures. A straight invariant fundamental cocycle $\tilde{\omega}$ is uniquely extended to a function on $\mathcal{M}^{n+1}$,

$$\begin{aligned}&\tilde{\omega}(\mu_0, \mu_1, \cdots, \mu_n) \\ &= \int_{\widetilde{M}^{n+1}} \tilde{\omega}(y'_0, y'_1, \cdots, y'_n) d\mu_0(y'_0) d\mu_1(y'_1) \cdots d\mu_n(y'_n).\end{aligned}$$

A smoothing operator is defined to be a smooth map

$$\mathscr{S} : \widetilde{M} \to \mathcal{P}.$$

<u>ℓ^∞-norm of straight invariant fundamental cocycle</u>

$$\|\tilde{\omega}\|_\infty = \sup \tilde{\omega}(y_0, y_1, \cdots, y_n) = \sup \tilde{\omega}(\mathscr{S}(y_0), \mathscr{S}(y_1), \cdots, \mathscr{S}(y_n)).$$

Theorem

If $\tilde{\omega}$ is a straight invariant fundamental cocycle, then $\mathscr{S}^*\tilde{\omega}$ defined by

$$\mathscr{S}^*\tilde{\omega}(y_0, y_1, \cdots, y_n) = \tilde{\omega}(\mathscr{S}(y_0), \cdots, \mathscr{S}(y_n))$$

is also a straight invariant fundamental cocycle.

Proposition

$$\|M\|' = \frac{1}{\inf \|\mathscr{S}^*\tilde{\omega}\|_\infty},$$

where the infimum is taken over all straight invariant fundamental cocycles $\tilde{\omega}$.

Motivation for the definition of smoothing operator

Let δ_y be the Dirac function at $y \in \widetilde{M}$, $\tilde{\omega}$ is any straight invariant fundamental cocycle on $\widetilde{M}$.

Define the operator $\mathscr{S}_\delta : \widetilde{M} \to \mathcal{M}$,

$$\mathscr{S}_\delta(y) = \delta_y, \quad y \in \widetilde{M}.$$

According to definition,

$$\mathscr{S}_\delta^* \tilde{\omega} = \tilde{\omega}.$$

Hence $\tilde{\omega}$ can be identified with $\mathscr{S}_\delta^* \tilde{\omega}$.

The smoothing operator can be viewed as a replacement of Dirac measure by probability measure.

Main Idea for the proof of Gromov's smoothing inequality

Theorem

Let $(M, \mathcal{G})$ be a closed n-dimensional Riemannian manifold, and $\mathscr{S} : \widetilde{M} \to \mathcal{P}$ be a smoothing operator. Then the simplicial volume satisfies

$$\|M\| \leqslant n! \|d\mathscr{S}\|_{\infty}^{n} \operatorname{Vol}_{\mathcal{G}}(M).$$

Recall:

$\mathcal{M}$: Banach space of finite measures on $\widetilde{M}$

$\mathcal{P} \subset \mathcal{M}$: the subspace of probability measures

$\|d_y\mathscr{S}\|_\infty = \sup |d_y\mathscr{S}(\tau)|$, where the supremum is taken over all $\tau \in S_y$, $S_y \subset T_y\widetilde{M}$ is the unit tangent sphere.

$\|d\mathscr{S}\|_\infty = \sup_y \|d_y\mathscr{S}\|_\infty$.

Main idea of the proof:

Let $\pi : \widetilde{M} \to M$ be the universal covering. Fix a straight invariant fundamental cocycle $\tilde{\omega} \in C(\widetilde{M}, \mathbb{R})$.

(1). The smoothing operator $\mathscr{S} : \widetilde{M} \to \mathcal{P}$ induces a differential n-form $\tilde{\alpha}$ on $\widetilde{M}$:

$$\tilde{\alpha}_y(u_1, \cdots, u_n) = \tilde{\omega}(\mathscr{S}(y), d_y\mathscr{S}(u_1), \cdots, d_y\mathscr{S}(u_n)),$$

where $(u_1, \cdots, u_n) \in T_y\widetilde{M}$.

(2). Sup-norm of the differential form $\tilde{\alpha}$ satisfies

$$\|\tilde{\alpha}\|_\infty \leqslant n!\|\tilde{\omega}\|_\infty \|d\mathscr{S}\|_\infty^n.$$

The sup-norm of $\tilde{\alpha}$ is

$$\|\tilde{\alpha}\|_\infty = \sup \tilde{\alpha}_y(u_1, \cdots, u_n),$$

where the supremum is taken over all $y \in \widetilde{M}$ and unit tangent vectors $(u_1, u_2, \cdots, u_n) \in S_y \subset T_y\widetilde{M}$.

(3). The n-form $\tilde{\alpha}$ induces a differential n-form α on M, which is in the fundamental cohomology class $\Omega_M \in H^n(M;\mathbb{R})$. Hence, there holds

$$1 = \int_M \alpha \leqslant \|\tilde{\alpha}\|_\infty \mathrm{Vol}_{\mathcal{G}}(M) \leqslant n! \|\tilde{\omega}\|_\infty \|d\mathscr{S}\|_\infty^n \mathrm{Vol}_{\mathcal{G}}(M).$$

The smoothing inequality is obtained by taking infimum over all straight invariant fundamental cocycles $\tilde{\omega}$.

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Thank you!