What is stable commutator length?

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What is ..? Seminar in Simplicial Volume and Bounded Cohomology

14. 12. 2020

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Three ways to stumble upon scl:

1. Algebraic: Via Commutator length
2. Topologic: Via Surfaces
3. Analytic: Via Quasimorphisms

SCL: Algebraic

Set	Group G
Element	$g \in [G, G]$
Invariants	
Example	

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Example	$G = F_2, g = [a, b]$ $cl([a, b]) = 1$ $cl([a, b]^3) = 2$ $cl([a, b]^n) = \lceil \frac{n+1}{2} \rceil$ $scl([a, b]) = \frac{1}{2}$

SCL: Geometric

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Element	$\gamma: S^1 \to X$ $\gamma \in [\pi_1(X), \pi_1(X)]$
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Example	$X = \Sigma_{1,1} =$  $\gamma = \partial\Sigma_{1,1}$ $\Phi = id : \Sigma_{1,1} \to \mathrm{X}$ $scl'(\gamma) := \inf \frac{-\chi(\Sigma)}{2\, n(\Phi)} \leq -\frac{-1}{2} = \frac{1}{2}$

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SCL: Analytic

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Invariants	Maps $\phi: G \to$ R, such that there is a $C > 0$: $\forall g, h \in G \ \lvert \phi(g) + \phi(h) - \phi(gh) \rvert < C$ Smallest such C: $D(\phi)$ $scl''(g) := \sup \frac{\phi(g)}{2\,D(\phi)}$ Where sup runs over all homogenous QM
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Example	$G = F_2, g = [a, b]$ $\phi = \phi_1 - \phi_2$ ϕ_1*: count subword ab;* ϕ_2*: count subword ba.* $D(\phi) = 2$ $\phi([a, b]) = 2$ $scl''(g) := \sup \frac{\phi(g)}{2\, D(\phi)} \geq \frac{2}{4} = \frac{1}{2}$

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Relationship to:

- Bounded Cohomology
- Circle Actions
- Combinatorics of words
- …

… of course

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If G is the fundamental group associated to X and g corresponds to γ, then

$$scl(g) = scl'(\gamma) = scl''(g)$$

[Calegari + Bavard]

Basic Properties

- **Linear:** $\forall g \in G: scl(g^n) = n \cdot scl(g)$
- **Quasi-Length:** $\forall g, h \in G: scl(g \cdot h) \leq scl(g) + scl(h) + \frac{1}{2}$

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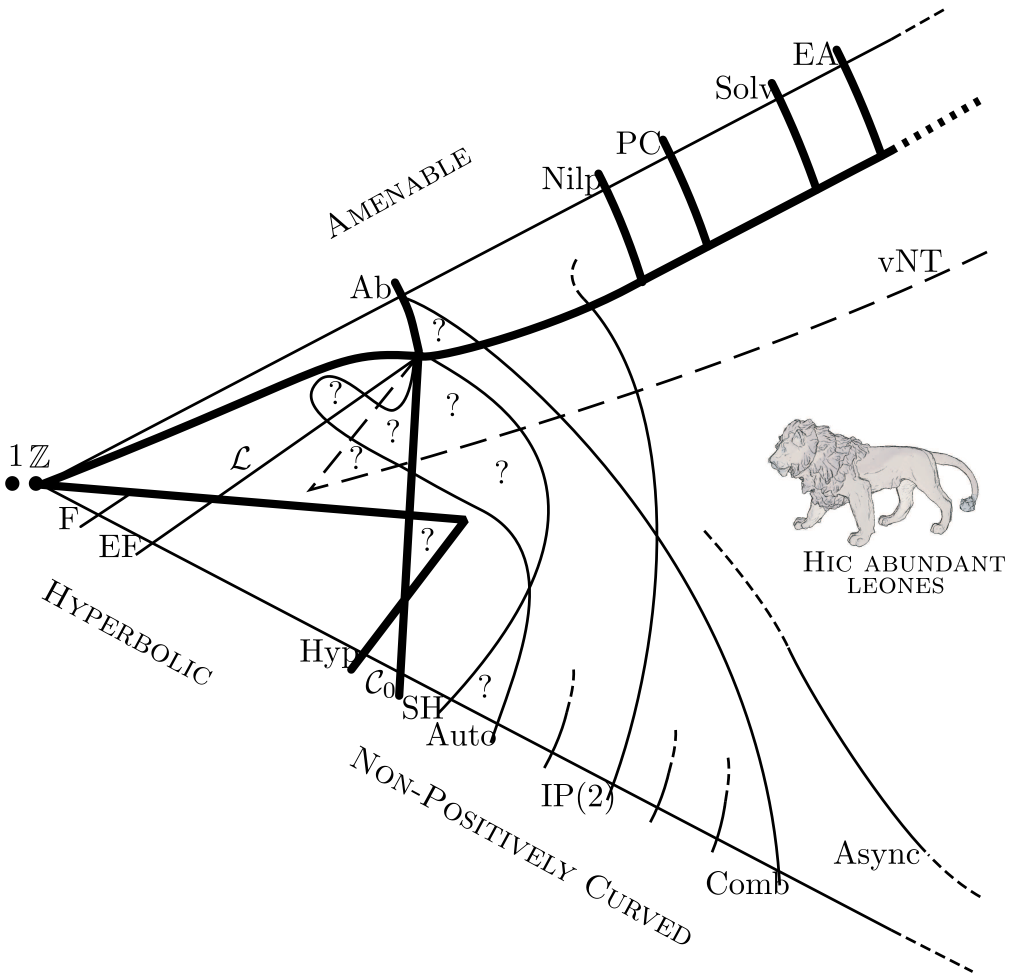
  Thus scl is invariant under automorphisms / retracts.
- **Finite index Subgroups:**
  If H < G is a finite index subgroup, then
  $$scl_H(g) = \frac{1}{[G:H]}\, scl_G\left(\sum_a a\, g\, a^{-1}\right)$$
  for a ranging over coclass representatives.

SCL on FP Groups

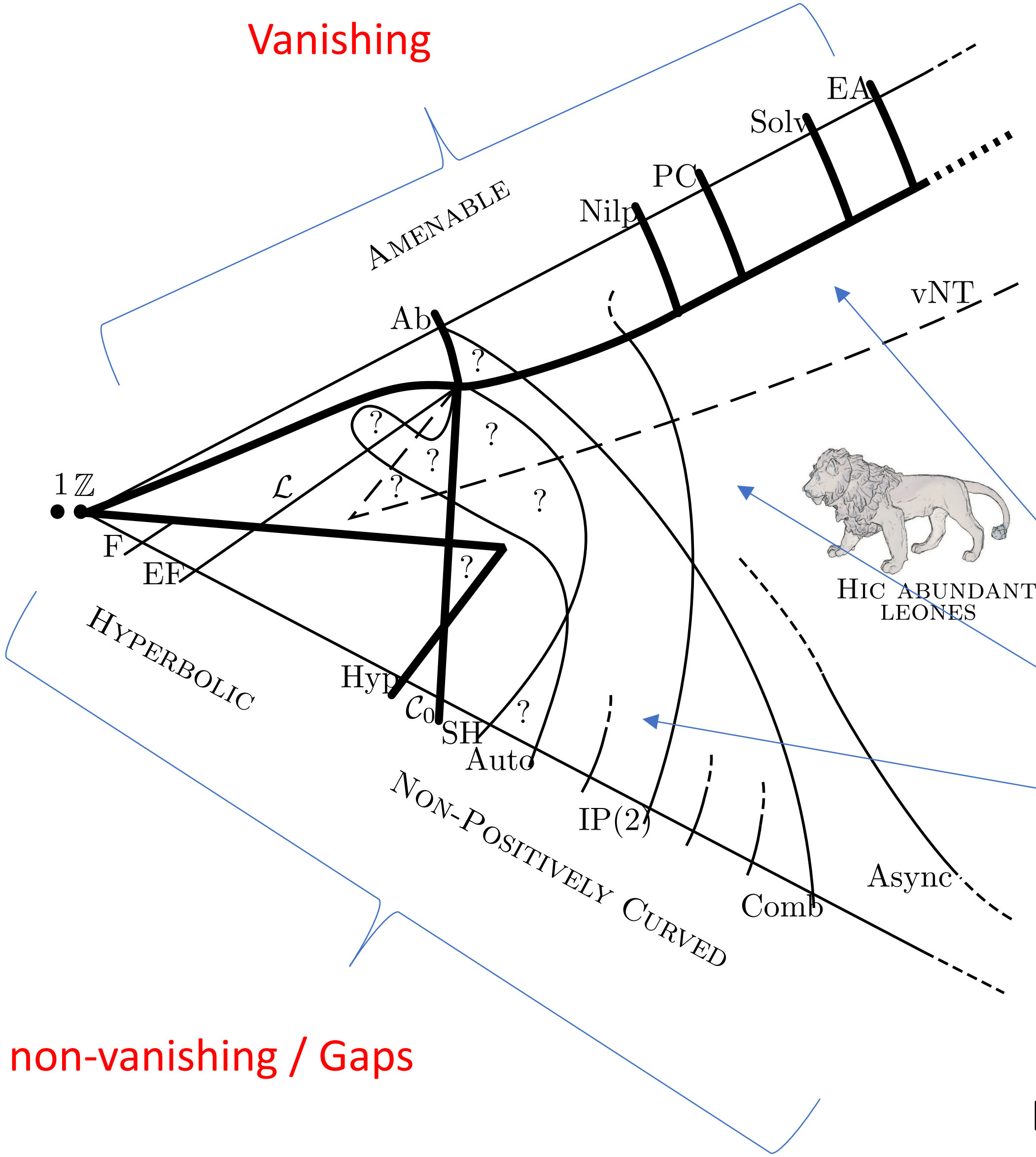


Bridon's Universe of FP Groups ([1])

SCL on FP Groups

Vanishing

Undecidability



Groups with an interesting spectrum

Strict non-vanishing / Gaps

Bridon's Universe of FP Groups ([1])

Vanishing

G satifies that $scl(g) = 0 \; \forall g \in G$ for:

- G amenable
- Piecewise linear Transformations of Interval (Calegari)
- Thompson's Group T

Non-Vanishing: Gaps

G has a gap in scl if there is a $C > 0$ such that for all but 'controlled' elements g, we have that $scl(g) \geq C$.

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Why useful?

Suppose that H arises as a finite index subgroup in G but the index is unknow.

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Why useful?

Suppose that H arises as a finite index subgroup in G but the index is unknow. Then, using

$$scl_G(g) = \frac{1}{[G:H]} \, scl_H(\textstyle\sum_a a \, g \, a^{-1})$$

we can bound the index $[G:H]$ from below.

Non-Vanishing: Gaps

G has a gap in scl if there is a $C > 0$ such that for all but 'controlled' elements g, we have that $scl(g) \geq C$.

For a big class of groups:

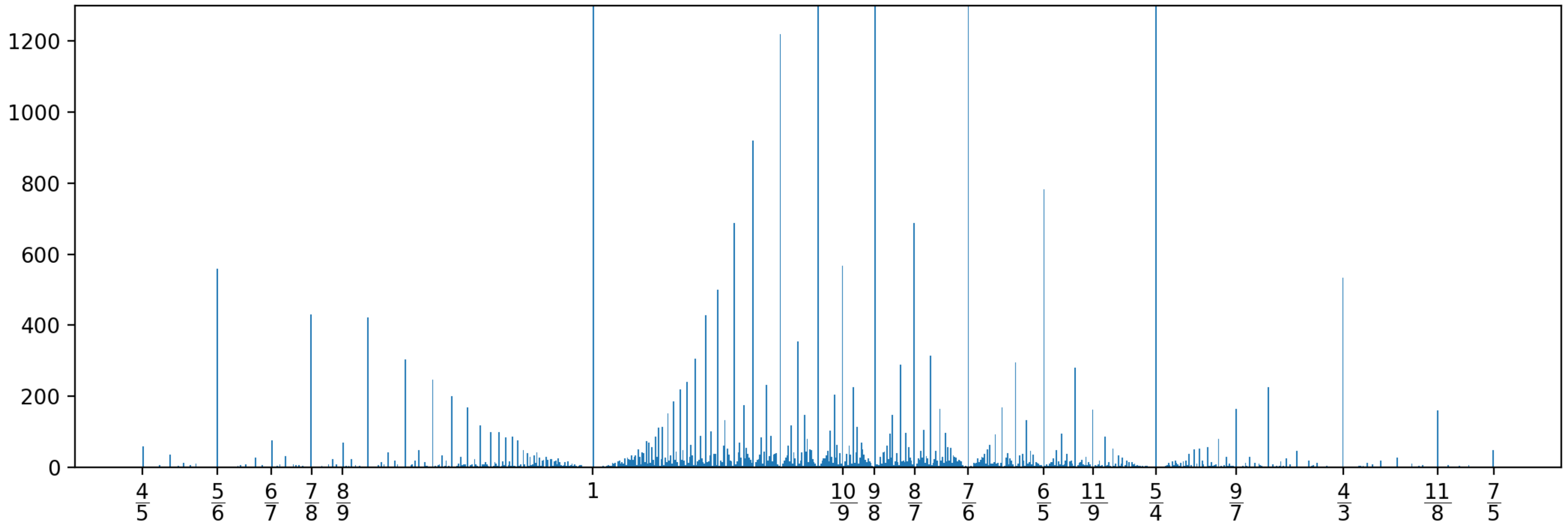
- Free groups (Duncan-Howie)
- Hyperbolic groups (Fujiwara Kapovich)
- Mapping Class Groups (Bestvina-Bromberg-Fujiwara)
- 3-manifold groups (Chen-H.)
- Certain Amalgamated Free Products (Chen-H., Clay-Forester-Louwsma)
- RAAGS (H., Forester-Tao-Soroko)

Decidability

Proposition: It is undecidable if an element $g \in G$ has vanishing scl or not.

Spectrum

- Free Groups: Have rational scl
  +there is a fast algorithm to compute it (Calegari, Calegari-Walker) Figure: 50.000 random elements of length 24 in F_2.



Spectrum

- Free Groups: Have rational scl (Calegari)
- BS groups have rational scl (Chen)
- One of the few groups, where full scl-spectrum is known: Universal Central Extension of Thompson’s Group T: Has scl all non-negative rationals
- There are groups with non-rational scl (Zhuang)

- For recursively finite groups: all right-computable numbers (H.)

Links to other fields: Simplicial Volume

Theorem (H. – Löh):
Let G be a fp group with $H_2(G; R) = 0$ and let $g \in [G, G]$ be an element. Then there a 4-manifold M such that

$$|| M || = 48 \cdot scl(g).$$

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Corollaries:

- There are 4-manifolds with arbitrary rational simplicial volume
- The set of simplicial volumes in higher dimensions is dense.

Open Questions

- What are extremal quasimorphisms for arbitrary elements of the free group?
- Is there a second gap of scl in non-abelian free groups F between $\frac{1}{2}$ and $\frac{7}{12}$
- Is there a finitely presented group which has algebraic but not rational values scl? Is the set of scls on finitely presented groups the set of right-computable numbers?
- Is scl rational on surface groups? If yes, is this rationality achieved using extremal surfaces? What about scl on Gromov hyperbolic groups?

Thank you for listening!