Simplicial Volume of Fiber Bundles with $K \le 0$ Fibers

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for two M^m, N^n occ mflds

oriented closed connected

$$\|M\| \cdot \|N\| \le \|M \times N\| \le \binom{n+m}{n} \|M\| \cdot \|N\|$$

Q: generalize these result to fiber bundle?

Example (Hoster and Kotschick, 2001)

$\exists$ $M \hookrightarrow E^3 \to S^1$ with E closed hyperbolic

Q: $\|E\| \leftarrow M, B$

Notation: $M \hookrightarrow E \xrightarrow{p} B$

Thm (Gromov, 82)

M amenable $\Rightarrow$ $\|E\| = 0$

Thm (Hoster and Koschick, 2001)

M surface $\Rightarrow$ $\|E\| \ge \|M\| \cdot \|B\|$

$\dim(E) \le 4$ $\Rightarrow$

Thm (Bucher, 2008)

M surface $\quad \dim E = p$

$$\|E\| \ge \begin{cases} 2^{\frac{p-1}{p}} \|M\| \cdot \|B\| & p \text{ even} \\ \frac{2p}{p+1} \|M\| \cdot \|B\| & p \text{ odd.} \end{cases}$$

Thm (Löh and Moraschini, 2021)

$$\mathrm{cat}_{Am}(M) \le \frac{\dim(E)}{\dim(B)+1} \Rightarrow \|E\| = 0$$

amenable category

Thm (Kastenholz and Reinhold, 2021)

$B = S^d$, $d \ge 2 \Rightarrow \|E\| = 0$.

Simplicial volume closely related to curvatures.

Thm 1: M, $K \le 0$, Ricci < 0

$\tilde{B} \to B$ universal covering, with $\tilde{B}$ closed

$\Rightarrow \|E\| = 0$

Thm 2 $\quad M \quad K < 0$, $\dim M > 2$

$\Rightarrow \quad \|E\| = 0 \Leftrightarrow \|B\| = 0$

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apply method in [Farrell, Gogolev, 2016]

the Center Thm:

M, $K \le 0$, $\Rightarrow$ $\mathrm{Center}(\pi_1(M)) = \mathbb{Z}^k$ for some $k \in \mathbb{N}$.

and $\exists$ covering $T^k \times M' \to M$ with M', $K \le 0$.

Cor: M, $K \le 0$, Ricci < 0 $\Rightarrow$ $\mathrm{Center}(\pi_1(M)) = \{e\}$

Def: (fiber homotopically trivial bundle)

if $\exists q: E \to M$ continuous

s.t. $q|_{M_x}: M_x \xrightarrow{\simeq} M$, for $\forall x \in B$

$M_x = p^{-1}(x)$ homotopy equivalence

Note: $\Rightarrow$ $E \simeq M \times B$

$\Rightarrow$ $\|E\| = \|M \times B\|$

Pf of Thm 1:

assume $\pi_1(B)=\{e\}$

$$\left(\begin{array}{ccc} \pi^*E & \to & E \\ \downarrow & & \downarrow \\ \tilde{B} & \xrightarrow{\pi} & B \end{array}\right)$$

triangulation $K \xrightarrow{r} B$

CW-complex

↳ pull-back bundle:

$$\begin{array}{ccc} r^*E & \to & E \\ \downarrow & & \downarrow \\ K & \xrightarrow{r} & B \end{array}$$

$K^1 \overset{\sigma}{\hookrightarrow} K$ 1 skeleton.

$\xRightarrow{\pi_1(K)=\{e\}}$ $\exists\ c: K^1 \to \{x_0\} \in K$ constant map

s.t. $\sigma \overset{H}{\simeq} c$

$\xRightarrow[\text{thm}]{\text{covering homotopy}}$ $\exists \tilde{H}: p^{-1}(K^1)\times[0,1] \to r^*E$

Cover H

note that $\tilde{H}(\cdot,1): p^{-1}(K^1) \to (r^*E)_{x_0}$

and $\tilde{H}(x,\cdot): p^{-1}(x)\times[0,1] \to r^*E$

$$p^{-1}(x) \simeq p^{-1}(x_0)$$
$$x \in K^1$$

extend $\tilde{H}(\cdot,1)$ inductively to successive skeleton of K.

$$\pi_n(G(M)) = \begin{cases} \mathrm{Center}(\pi_1(M))=\{e\} & n=1 \\ \{e\} & n\geq 2 \end{cases}$$

self homotopy equivalences of M

[Bustamante, Farrell, Jiang, 2016]

Def. (fiber homotopy equivalent).

$$\exists \quad \begin{array}{ccc} E_1 & \underset{\psi}{\overset{\varphi}{\rightleftarrows}} & E_2 \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

s.t. $\psi\circ\varphi \simeq Id_{E_1}$, $\varphi\circ\psi \simeq Id_{E_2}$

fiber preserving

Notation. $\mathrm{Diff}(M)=\{f: M\to M \text{ diffeo}\}$

$\mathrm{Diff}_0(M)=\{f\in \mathrm{Diff}(M) \mid f\simeq id_M\}$

G gp $\Rightarrow$ universal principal G-Bundle

$$\begin{array}{ccc} G & \to & EG \\ & & \downarrow \\ & & BG \end{array}$$

$\gamma: G\to G' \rightsquigarrow \bar{\gamma}: BG \to BG'$

weak homotopy $\Rightarrow$ ...

[Dold, Lashof, 1959]

Note. $\begin{array}{c} M \hookrightarrow E \\ \downarrow \\ B \end{array}$ $\underset{\text{homotopy}}{\overset{1:1}{\longleftrightarrow}}$ $E \to B\mathrm{Diff}(M)$

classifying map

Recall: X CW-complex

weak homotopy equivalence $Y\to Z$

$\rightsquigarrow [X,Y] \overset{1:1}{\longleftrightarrow} [X,Z]$

homotopy classes

pull-back bundle

$$\begin{tikzcd} B' \arrow[r] \arrow[d, "\theta"] & BDiff_0(M) \arrow[d] \\ B \arrow[r, "g"] & BDiff(M) \end{tikzcd}$$

classfying map.

$BDiff_0(M) = EG(M)/Diff_0(M)$

$Diff_0(M) < Diff(M) < G(M)$

$BDiff(M) = EG(M)/Diff(M)$.

$$\begin{tikzcd} \theta^*E \arrow[r] \arrow[d] & E \arrow[d] \\ B' \arrow[r, "\theta"] & B \end{tikzcd}$$

$$\begin{tikzcd} Diff(M)/Diff_0(M) \arrow[r] \arrow[d, leftrightarrow, "1:1"] & BDiff(M) \arrow[d] \\ \pi_0(G(M)) & BDiff(M). \end{tikzcd}$$

$\pi_0(G(M)) \cong Out(\pi_1(M))$ finite

$$B' \to BDiff_0(M) \to BDiff(M) \xrightarrow{\Phi} B_{G(M)} \to B_{\pi_0(G(M))}$$

weak homotopy equivalence

$\Uparrow$

$G(M) \to \pi_0(G(M))$
weak homotopy equivalence

Recall: $E_1 \simeq E_2 \Leftrightarrow \Phi \circ g_1 \simeq \Phi \circ g_2$

$\Phi : BDiff(M) \to BG(M)$.

$\Rightarrow \theta^*E \simeq M \times B$

$\square$