

Simplicial Volume of Fiber Bundles with  $K \leq 0$  Fibers

- Xiaofeng Meng

for two  $M^m, N^n$  occ manifolds

oriented closed connected

$$\|M\| \cdot \|N\| \leq \|M \times N\| \leq \binom{m+n}{n} \|M\| \cdot \|N\|$$

Q: generalize these result to fiber bundle?

Example (Hoster and Koschick, 2001)

$$\exists M \hookrightarrow E^3 \text{ with } E \text{ closed hyperbolic}$$

$$\downarrow$$

$$S^1$$

$$Q: \|E\| \leftarrow M, B$$

Notation:  $M \hookrightarrow E$

$$\downarrow p$$

$$B$$

Thm (Gromov, 82)

$$M \text{ amenable} \Rightarrow \|E\| = 0$$

Thm (Hoster and Koschick, 2001)

$$M \text{ surface} \Rightarrow \|E\| \geq \|M\| \cdot \|B\|$$

$$\dim(E) \leq 4 \Rightarrow$$

Thm (Bucher, 2009)

$$M \text{ surface } \dim E = p$$

$$\|E\| \geq \begin{cases} 2 \frac{p-1}{p} \|M\| \cdot \|B\| & p \text{ even} \\ \frac{2p}{p+1} \|M\| \cdot \|B\| & p \text{ odd} \end{cases}$$

Thm (Löh and Moraschini, 2021)

$$\text{Cat}_{\text{Am}}(M) \leq \frac{\dim(E)}{\dim(B)+1} \Rightarrow \|E\| = 0$$

amenable category

Thm (Kastanholz and Reinhold, 2021)

$$B = S^d, d \geq 2 \Rightarrow \|E\| = 0$$

Simplicial volume closely related to curvatures.

Thm 1:  $M, K \leq 0, \text{Ricci} < 0$

$\tilde{B} \rightarrow B$  universal covering, with  $\tilde{B}$  closed

$$\Rightarrow \|E\| = 0$$

Thm 2:  $M, K < 0, \dim M > 2$

$$\Rightarrow \|E\| = 0 \Leftrightarrow \|B\| = 0$$

apply method in [Farrell, Gogalev, 2016]

the Center thm

$$M, K \leq 0, \Rightarrow \text{Center}(\pi_1(M)) = \mathbb{Z}^k$$

for some  $k \in \mathbb{N}$ .

and  $\exists$  covering  $T^k \times M' \rightarrow M$  with  $M', K \leq 0$ .

Cor:  $M, K \leq 0, \text{Ricci} < 0 \Rightarrow$

$$\text{Center}(\pi_1(M)) = \{e\}$$

Def: (fiber homotopically trivial bundle)

if  $\exists f: E \rightarrow M$  continuous

s.t.  $f|_{M_x}: M_x \xrightarrow{\cong} M$ , for  $\forall x \in B$

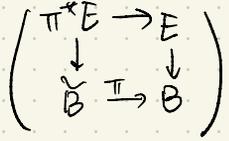
"  $p^{-1}(x)$  homotopy equivalence

Note:  $\hookrightarrow E \subseteq M \times B$

$$\Rightarrow \|E\| = \|M \times B\|$$

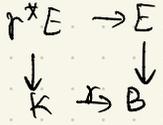
pf of Thm 1:

assume  $\pi_1(B) = \{e\}$ .



triangulation  $K \xrightarrow{r} B$   
 $\uparrow$   
 CW-complex

$\hookrightarrow$  pull-back bundle:



$K^1 \xrightarrow{\sigma} K$  1 skeleton.

$\xrightarrow{\pi_1(K) = \{e\}}$   $\exists c: K^1 \rightarrow \{x_0\} \in K$  constant map  
 s.t.  $\sigma \stackrel{H}{\simeq} c$

$\xrightarrow[\text{then}]{\text{covering homotopy}}$   $\exists \tilde{H}: \tilde{p}^{-1}(K^1) \times [0,1] \rightarrow r^*E$

cover  $H$

note that  $\tilde{H}(\cdot, 1): \tilde{p}^{-1}(K^1) \rightarrow (r^*E)_{x_0}$

and  $\tilde{H}(x, \cdot): \tilde{p}^{-1}(x) \times [0,1] \rightarrow r^*E$

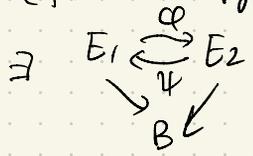
$$\tilde{p}^{-1}(x) \cong \tilde{p}^{-1}(x_0)$$

extend  $\tilde{H}(\cdot, 1)$  inductively to successive skeleton of  $K$ .

$\uparrow$   
 $\pi_n(G(M)) = \begin{cases} \text{Center}(\pi_1(M)) = \{e\} & n=1 \\ \{e\} & n \geq 2 \end{cases}$   
 self homotopy equivalences of  $M$

[Bustamante, Farrell, Jiang, 2016]

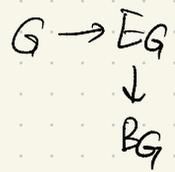
Def. (fiber homotopy equivalent)



s.t.  $\psi \circ \varphi \simeq \text{Id}_{E_1}$ ,  $\varphi \circ \chi \simeq \text{Id}_{E_2}$   
 $\uparrow$  fiber preserving

Notation:  $\text{Diff}(M) = \{f: M \rightarrow M \text{ diffeo}\}$   
 $\text{Diff}_*(M) = \{f \in \text{Diff}(M) \mid f \simeq \text{id}_M\}$

$G$  gp  $\Rightarrow$  universal principal  $G$ -Bundle



$\gamma: G \rightarrow G' \rightsquigarrow \bar{\gamma}: BG \rightarrow BG'$

weak homotopy  $\Rightarrow$

[Dold, Lashof, 1959]

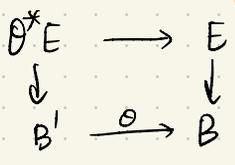
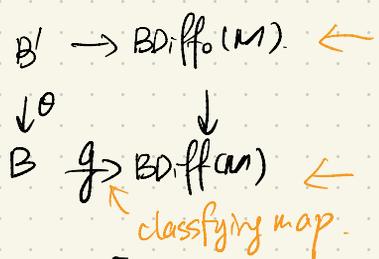
Note:  $M \hookrightarrow E \xrightarrow{\cong} E \rightarrow B\text{Diff}(M)$   
 $\downarrow$   $B$   $\xrightarrow{\text{homotopy}}$   $\uparrow$   $\text{classifying map}$

Recall:  $X$  CW-complex

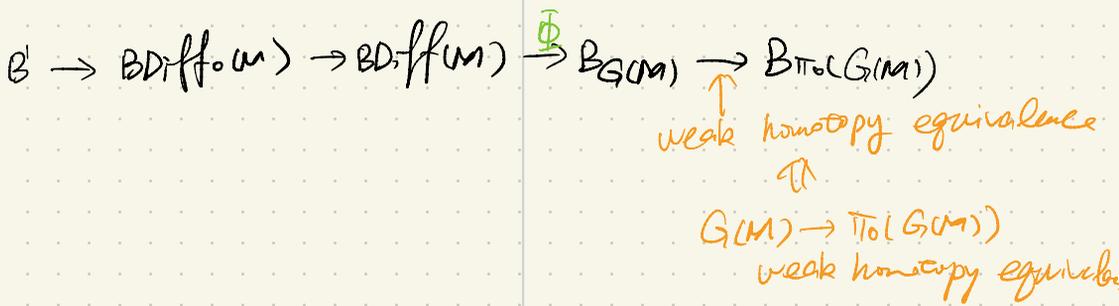
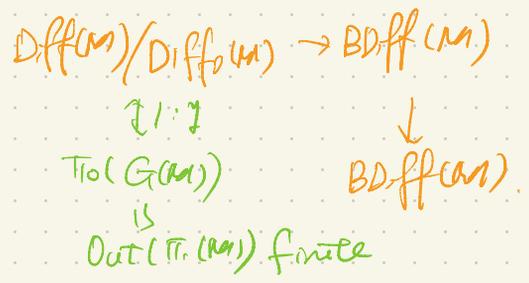
weak homotopy equivalence  $\gamma \rightarrow \gamma'$

$\Rightarrow [X, \gamma] \xrightarrow{\cong} [X, \gamma']$   
 $\uparrow$   
 homotopy classes

pull-back bundle



$$\begin{aligned}
 B\text{Diff}_0(M) &= \text{EG}(M)/\text{Diff}_0(M) \\
 \text{Diff}_0(M) &< \text{Diff}(M) < G(M) \\
 B\text{Diff}(M) &= \text{EG}(M)/\text{Diff}(M)
 \end{aligned}$$



Recall:  $E_1 \cong E_2 \iff \Phi \circ \varphi_1 \simeq \Phi \circ \varphi_2$

$$\Phi: B\text{Diff}(M) \rightarrow B_{G(M)}$$

$$\Rightarrow \theta^* E \simeq M \times B$$

□