Higher-degree bounded cohomology of transformation groups

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Motivation and results

We consider the following (discrete) transformation groups:

- $\mathrm{Homeo}_{\mathrm{vol},0}(M)$, the volume-preserving, isotopic-to-identity homeomorphisms on a compact Riemannian manifold M of dimension ≥ 3
- $\mathrm{Diff}_{\mathrm{vol}}(\mathbb{D}^2, \partial\mathbb{D}^2)$, the volume-preserving (smooth) diffeomorphisms on the standard 2–disk that restrict to the identity in a neighborhood of the boundary

Theorem (Brandenbursky–Marcinkowski, '19)

If the fundamental group $\pi_1(M)$ surjects onto the free group F_2, then the exact reduced bounded cohomology $\overline{\mathrm{EH}}_{\mathrm{b}}^{d}(\mathrm{Homeo}_{\mathrm{vol},0}(M))$ is infinite-dimensional in degrees $d \in \{2, 3\}$.

Theorem (Kimura, '20)

$\dim \overline{\mathrm{EH}}_{\mathrm{b}}^{d}(\mathrm{Diff}_{\mathrm{vol}}(\mathbb{D}^2, \partial\mathbb{D}^2)) = \infty$ in degrees $d \in \{2, 3\}$.

Main results

- If $\pi_1(M)$ surjects onto F_2, then $\dim \overline{\mathrm{EH}}_{\mathrm{b}}^{d}(\mathrm{Homeo}_{\mathrm{vol},0}(M)) = \infty$ for $d \geq 2$ even.
- $\dim \overline{\mathrm{EH}}_{\mathrm{b}}^{d}(\mathrm{Diff}_{\mathrm{vol}}(\mathbb{D}^2, \partial\mathbb{D}^2)) = \infty$ for $d \geq 2$ even.
- Assume that $\dim M \geq 5$ and that there exists a split surjection $\pi_1(M) \to \mathbb{Z}^2$ that is trivial on the center $Z(\pi_1(M))$. Then $\mathrm{H}^d(\mathrm{Homeo}_{\mathrm{vol},0}(M)) \neq \{0\}$ for all $d \geq 0$.

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Other groups with infinite bounded-cohomological dimension:

- $\bigoplus_{\mathbb{N}} \mathrm{F}_2$ and similar infinite direct sums (Löh, '15)
- certain finitely generated groups (Fournier-Facio–Löh–Moraschini, '21)
- $\mathrm{Homeo}_0(\mathbb{S}^1)$ and $\mathrm{Homeo}_0(\mathbb{D}^2)$ (Monod–Nariman, '21)

Argument outline

Theorem (Brandenbursky–Marcinkowski, '19)

If the fundamental group $\pi_1(M)$ surjects onto the free group F_2, then the exact reduced bounded cohomology $\overline{\mathrm{EH}}_{\mathrm{b}}^{d}(\mathrm{Homeo}_{\mathrm{vol},0}(M))$ is infinite-dimensional in degrees $d \in \{2, 3\}$.

Idea: We construct

$$\overline{\mathrm{EH}}_{\mathrm{b}}(\mathrm{F}_2) \xleftarrow{\quad {\alpha_I}^* \quad} \overline{\mathrm{EH}}_{\mathrm{b}}(\mathrm{Homeo}_{\mathrm{vol},0}(M)) \xleftarrow{\quad \mathbf{tr} \quad} \overline{\mathrm{EH}}_{\mathrm{b}}(\Gamma) \ ,$$

where Γ has non-trivial bounded cohomology, the ${\alpha_I}^*$ are induced by homomorphisms $\alpha_I \colon \mathrm{F}_2 \to \mathrm{Homeo}_{\mathrm{vol},0}(M)$, and $\mathbf{tr} \circ \alpha_I$ converges to something that can be computed.

For higher degrees do the same with

$$\overline{\mathrm{EH}}_{\mathrm{b}}({\mathrm{F}_2}^n) \xleftarrow{(\alpha_{1,I} \times \cdots \times \alpha_{n,I})^*} \overline{\mathrm{EH}}_{\mathrm{b}}(\mathrm{Homeo}_{\mathrm{vol},0}(M)) \xleftarrow{\quad \mathbf{tr}' \quad} \overline{\mathrm{EH}}_{\mathrm{b}}(\Gamma^n) \ .$$

The induction map

Definition (coupling between groups)

A *coupling* between discrete groups Γ, Λ is a measure space X with measure-preserving commuting actions $\sigma\colon \Gamma \curvearrowright X$ and $\rho\colon X \curvearrowleft \Lambda$.
The coupling is *left-cofinite* if Γ is countable σ is free and there exists a finite Γ-fundamental domain.

Main example

$X = \widetilde{M}$ the universal covering of a compact Riemannian manifold. $\Gamma := \pi_1(M)$ acts by deck transformation and Λ by measure-preserving Γ–equivariant homeomorphisms.

A choice of a fundamental domain F gives rise to an isomorphism $X \cong \Gamma \times F$.
Let $\chi := pr_\Gamma\colon X \cong \Gamma \times F \to \Gamma$. We define the chain map (as in Monod–Shalom, '06)

$$\chi^*\colon \ell^\infty(\Gamma^{n+1};\mathbb{R})^\Gamma \to \ell^\infty\left(\Lambda^{n+1};\mathcal{L}^\infty(X,\mathbb{R})^\Gamma\right)^\Lambda$$

$$\chi^* c(\lambda_0,\ldots,\lambda_n)(x) := c\Big(\chi(x.\lambda_0),\ldots,\chi(x.\lambda_n)\Big).$$

$$\Gamma \curvearrowright \ell^\infty(\Gamma^{n+1};\mathbb{R})\colon (\gamma.c)(\gamma_0,\ldots,\gamma_n) = \gamma.c(\gamma^{-1}\gamma_0,\ldots,\gamma^{-1}\gamma_n)$$

$$\Lambda \curvearrowright \ell^\infty(\Lambda^{n+1};\widehat{E})\colon (\lambda.c)(\lambda_0,\ldots,\lambda_n) = \lambda.c(\lambda^{-1}\lambda_0,\ldots,\lambda^{-1}\lambda_n)$$

$$\Gamma \curvearrowright \mathcal{L}^\infty(X,\mathbb{R})\colon (\gamma.f)(x) = \gamma.f(\gamma^{-1}.x)$$

$$\Lambda \curvearrowright \mathcal{L}^\infty(X,\mathbb{R})\colon (\lambda.f)(x) = f(x.\lambda)$$

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This gives rise to a homomorphism

$$\mathbf{ind}_\Gamma^\Lambda X\colon \qquad \mathrm{H}_\mathrm{b}^*(\Gamma;\mathbb{R}) \to \mathrm{H}_\mathrm{b}^*(\Lambda;\mathcal{L}^\infty(X,\mathbb{R})^\Gamma).$$

The transfer map

For a choice of a fundamental domain F let $\chi := pr_\Gamma \colon X \cong \Gamma \times F \to \Gamma$.

$$\mathbf{ind}_\Gamma^\Lambda X\colon \quad \mathrm{H}_\mathrm{b}^*(\Gamma;\mathbb{R}) \to \mathrm{H}_\mathrm{b}^*(\Lambda; \mathcal{L}^\infty(X,\mathbb{R})^\Gamma)$$

$$\chi^*\colon \ell^\infty(\Gamma^{n+1};\mathbb{R})^\Gamma \to \ell^\infty\left(\Lambda^{n+1}; \mathcal{L}^\infty(X,\mathbb{R})^\Gamma\right)^\Lambda$$

$$\chi^* c(\lambda_0,\dots,\lambda_n)(x) := c\big(\chi(x.\lambda_0),\dots,\chi(x.\lambda_n)\big)$$

Lemma

The induction homomorphism does not depend on the choice of the fundamental domain.

Idea: if $F' \neq F$ is a different fundamental domain, then

$$\mathbf{ind}_\Gamma^\Lambda(X,F) = (\text{right-multiplication}_\Delta)^* \circ \mathbf{ind}_\Gamma^\Lambda(X,F')$$

Finally, set $\quad \mathbf{tr}_\Gamma^\Lambda X\colon \quad \mathrm{H}_\mathrm{b}^*(\Gamma;\mathbb{R}) \xrightarrow{\mathbf{ind}} \mathrm{H}_\mathrm{b}^*(\Lambda; \mathcal{L}^\infty(X,\mathbb{R})^\Gamma) \xrightarrow{\int_*} \mathrm{H}_\mathrm{b}^*(\Lambda;\mathbb{R})$

$$[c] \mapsto \left[(\lambda_0,\dots,\lambda_n) \mapsto \int_F c\big(\chi(x.\lambda_0),\dots,\chi(x.\lambda_n)\big)\right]$$

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$$[c] \mapsto \left[(\lambda_0,\dots,\lambda_n) \mapsto \int_F c\Big(\chi(x.\lambda_0),\dots,\chi(x.\lambda_n)\Big)\right]$$

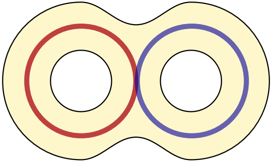
It is easy to check that the transfer satisfies the following properties:

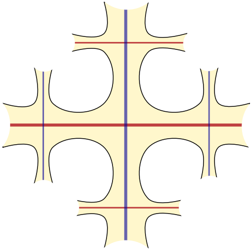
1. If $X=\Gamma$ with the counting measure, then Λ acts by $x.\lambda = x\cdot\varphi(\lambda)$ for some φ and $\mathbf{tr}_\Gamma^\Lambda X = \varphi^*$. (group morphisms)
2. $\mathbf{tr}_\Gamma^\Lambda(X \sqcup X') = \mathbf{tr}_\Gamma^\Lambda X + \mathbf{tr}_\Gamma^\Lambda X'$ and (linearity)
   $\mathbf{tr}_\Gamma^\Lambda(X \times X_{\mathrm{tr}}) = \mathrm{vol}(X_{\mathrm{tr}}) \cdot \mathbf{tr}_\Gamma^\Lambda X$ if X_{tr} has trivial Γ– and Λ–actions
3. $\mathbf{tr}_\Gamma^\Pi(X_1 \times_\Lambda X_2) = \mathbf{tr}_\Lambda^\Pi X_2 \circ \mathbf{tr}_\Gamma^\Lambda X_1$ and (concatenation)
   $\mathbf{tr}_{\Gamma,\sigma}^{\Pi,\rho\circ\varphi} X = \varphi^* \circ \mathbf{tr}_{\Gamma,\sigma}^{\Lambda,\rho} X$
4. $\mathbf{tr}_{\Gamma_1\times\Gamma_2}^{\Lambda_1\times\Lambda_2}(X_1 \times X_2)(\xi_1 \times \xi_2) = \mathbf{tr}_{\Gamma_1}^{\Lambda_1} X_1(\xi_1) \times \mathbf{tr}_{\Gamma_2}^{\Lambda_2} X_2(\xi_2)$ (products)
5. If $\Big((\Gamma,\sigma,X,\rho_l,\Lambda)\Big)_{l\in\mathbb{N}}$ is a sequence of left-cofinite couplings and $\rho_l \to \rho_\infty$ in the sense that $\mathrm{vol}\Big(F \cap \bigcup_{\lambda\in\Lambda}\{\rho_l(\lambda)(x) \neq \rho_\infty(\lambda)(x)\}\Big) \to 0$, then (limits)
   $$\sup_{\|\xi\|=1} \Big\| \mathbf{tr}_{\Gamma,\sigma}^{\Lambda,\rho_l} X(\xi) - \mathbf{tr}_{\Gamma,\sigma}^{\Lambda,\rho_\infty} X(\xi) \Big\| \to 0.$$

Special case of Brandenbursky–Marcinkowski's theorem

If $\pi_1(M) = \mathrm{F}_2$, then $\overline{\mathrm{EH}}{}^d_{\mathrm{b}}(\mathrm{Homeo}_{\mathrm{vol},0}(M))$ is infinite-dimensional for $d \in \{2,3\}$.

Sketch: • Use the coupling $\big(\pi_1(M), \sigma, \widetilde{M}, \rho, \mathrm{Homeo}_{\mathrm{vol},0}(M)\big)$. $\rho\colon \widetilde{M} \curvearrowleft \mathrm{Homeo}_{\mathrm{vol},0}(M)$ is given by $x.\lambda := \widetilde{H}_1(x)$, with $\widetilde{H}\colon [0,1] \times \widetilde{M} \to \widetilde{M}$ the lift of any isotopy from id_M to λ.





• Define $\alpha_l\colon \mathrm{F}_2 \to \mathrm{Homeo}_{\mathrm{vol},0}(M)$: send the generators of F_2 to the end result of the "finger-push" homotopy around the red/blue tube, with increasing tightness as $l \to \infty$.

• $\overline{\mathrm{EH}}_{\mathrm{b}}(\mathrm{F}_2) \xleftarrow{{\alpha_l}^*} \overline{\mathrm{EH}}_{\mathrm{b}}(\mathrm{Homeo}_{\mathrm{vol},0}(M)) \xleftarrow{\mathbf{tr}} \overline{\mathrm{EH}}_{\mathrm{b}}(\pi_1(M))$

Decompose the limit coupling $\big(\pi_1(M), \sigma, \widetilde{M}, \lim(\rho \circ \alpha_l), \mathrm{F}_2\big)$ into pieces: red, blue, intersection, the rest. Only the transfer of the intersection piece is non-trivial.

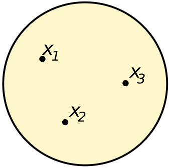
• The limit transfer map is a multiple of the identity on $\overline{\mathrm{EH}}_{\mathrm{b}}(\mathrm{F}_2)$.

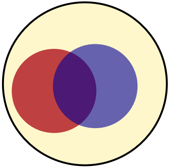
Hence $\mathbf{tr}\,\widetilde{M}\colon \overline{\mathrm{EH}}_{\mathrm{b}}(\pi_1(M)) \to \overline{\mathrm{EH}}_{\mathrm{b}}(\mathrm{Homeo}_{\mathrm{vol},0}(M))$ is injective.

Kimura's theorem

$\overline{\mathrm{EH}}_{\mathrm{b}}^{d}(\mathrm{Diff}_{\mathrm{vol}}(\mathbb{D}^2, \partial\mathbb{D}^2))$ is infinite-dimensional for $d \in \{2, 3\}$.

Sketch: • Let C_3 be the configuration space $\left\{(x_1, x_2, x_3) \in (\mathbb{D}^2)^3 \;\middle|\; x_i \neq x_j \;\; \forall i \neq j\right\}$ and $\widetilde{M}$ the universal covering of its Fulton–MacPherson compactification M





• Use the coupling $\left(\pi_1(M), \sigma, \widetilde{M}, \rho, \mathrm{Diff}_{\mathrm{vol}}(\mathbb{D}^2, \partial\mathbb{D}^2)\right)$.
The right action $\rho\colon \widetilde{M} \curvearrowleft \mathrm{Diff}_{\mathrm{vol}}(\mathbb{D}^2, \partial\mathbb{D}^2)$ is given by $x.\lambda := \widetilde{H}_1(x)$, where $\widetilde{H}\colon [0,1] \times \widetilde{M} \to \widetilde{M}$ is the lift of the induced path of any isotopy from $id_{\mathbb{D}^2}$ to λ.

• $\pi_1(M) =$ the pure braid group on 3 strands $\cong \mathbb{Z} \times \mathrm{F}_2 = \langle z \rangle \times \langle \tau_1, \tau_2 \rangle$, where z rotates all 3 points around each other and τ_1, τ_2 rotates x_3 around x_1, x_2.

• Define $\alpha_l\colon \mathrm{F}_2 \to \mathrm{Diff}_{\mathrm{vol}}(\mathbb{D}^2, \partial\mathbb{D}^2)$: send the generators of F_2 to the end result of twisting the red/blue disk once, with increasing tightness as $l \to \infty$.

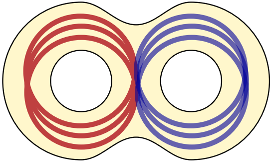
• As before compute the limit transfer map by decomposing the limit coupling.

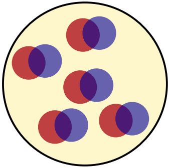
Ingredients:

1) Recall:

4. $\mathbf{tr}^{\Lambda_1 \times \Lambda_2}_{\Gamma_1 \times \Gamma_2}(X_1 \times X_2)(\xi_1 \times \xi_2) = \mathbf{tr}^{\Lambda_1}_{\Gamma_1} X_1(\xi_1) \times \mathbf{tr}^{\Lambda_2}_{\Gamma_2} X_2(\xi_2)$ (products)

2) We can take multiple (sequences of) group homomorphisms $\alpha_{i,l} \colon \mathrm{F}_2 \to \Lambda$ for $\Lambda = \mathrm{Homeo}_{\mathrm{vol},0}(M)$ or $\Lambda = \mathrm{Diff}_{\mathrm{vol}}(\mathbb{D}^2, \partial\mathbb{D}^2)$, with disjoint support:





• Consider the the sequence of homomorphisms $\alpha_l = \prod \alpha_{i,l} \colon {\mathrm{F}_2}^n \to \Lambda$ and the coupling $\left(\pi_1(M)^n, \sigma^n, (\widetilde{M})^n, \rho^n \circ \Delta_\Lambda, \Lambda\right)$:

$$\overline{\mathrm{EH}}_{\mathrm{b}}({\mathrm{F}_2}^n) \xleftarrow{(\alpha_{1,l} \times \cdots \times \alpha_{n,l})^*} \overline{\mathrm{EH}}_{\mathrm{b}}(\mathrm{Homeo}_{\mathrm{vol},0}(M)) \xleftarrow{\mathbf{tr}^{\rho^n \circ \Delta}_{\sigma^n}(\widetilde{M})^n} \overline{\mathrm{EH}}_{\mathrm{b}}(\pi_1(M)^n) \ ,$$

• Detect non-trivial elements on the left by pairing ℓ^1–homology. Note that the elements have the form $\bigcup_{i=1}^n \sum_{j=1}^n {q_j}^*(z_i)$ for $z_i \in \overline{\mathrm{EH}}_{\mathrm{b}}(\mathrm{F}_2)$ and $q_j \colon {\mathrm{F}_2}^n \to \mathrm{F}_2$ the j–th projection.