Algorithmic recognition of spatial graphs

joint with Stefan Friedl, Lars Munser & Yuri Santos Rego

Everything PL!

A spatial graph is a triple

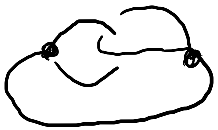
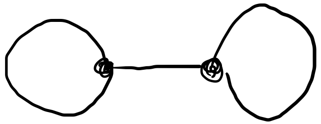
$$\Gamma = (V, E), \text{ with}$$

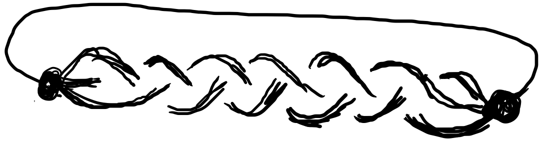
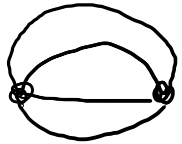
- $V \subset S^3$ finite subset "vertex set"
- E a finite set of PL-subspaces $e \subset S^3$

$$e \underset{PL}{\cong} \begin{cases} \text{arc} & e \cap V = \text{endpts of } e \\ \text{circle} & e \cap V = \text{singleton} \end{cases}$$

"edge set"

An isomorphism $\Gamma_1 \to \Gamma_2$ is an or.-pres. PL homeo $\Phi : S^3 \to S^3$ taking
| vertices of Γ_1 to vertices of Γ_2,
| edges —— edges ——

 $\not\cong$ 

 $\cong$ 

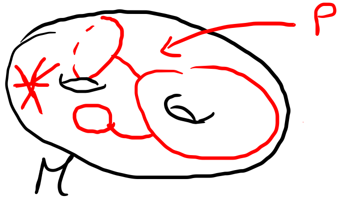
Main theorem. There is an algorithm to test if two spatial graphs are isomorphic.

Main tool: [Matveev, 2003]

There is an algorithm to test if two $\boxed{\text{Haken manifolds with boundary pattern}}$ are homeomorphic.

A manifold with boundary pattern is a pair (M, P), where

- M is a compact 3-manifold,
- $P \subset \partial M$ is a 1-dim subpolyhedron.

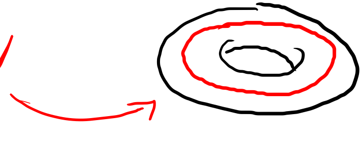


(M, P) is Haken if:

- M is "sufficiently large" ← contains "interesting" embedded surfaces
- M is irreducible: embedded 2-spheres bound balls
- (M, P) is ∂-irreducible: if a curve in ∂M away from P bounds a disc in M, it already bounds in ∂M.

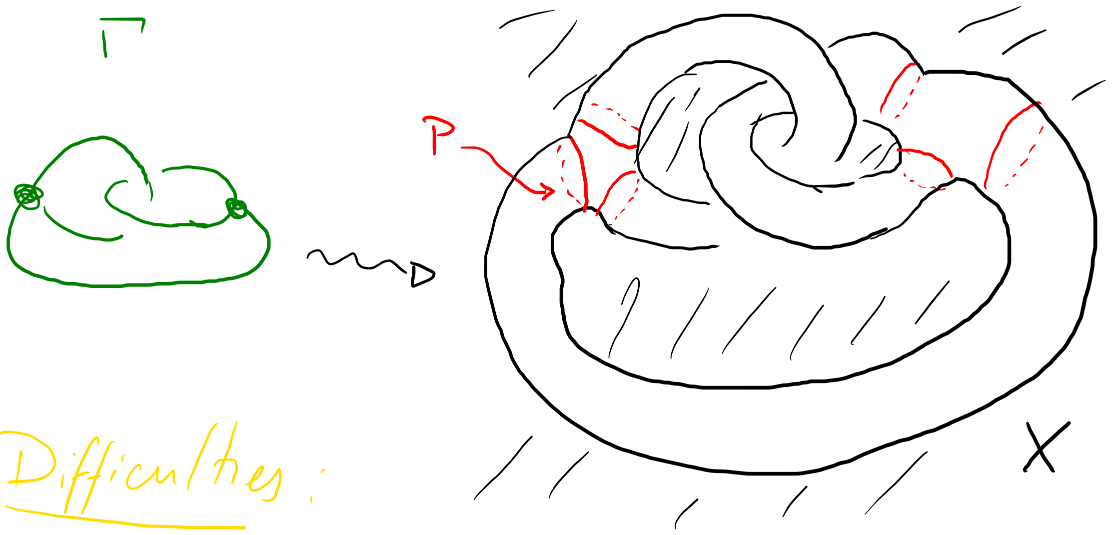
not ∂-irred

∂-irred



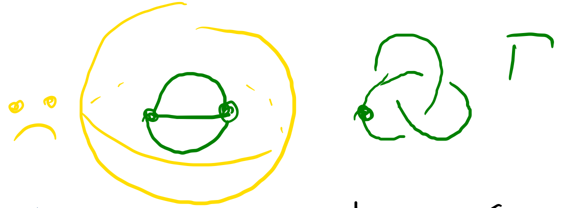
Idea: Given spatial graph Γ, mark boundary of $X := S^3 \setminus \nu(\Gamma)$ with P so that Γ is retrievable.

Then compare (X, P).

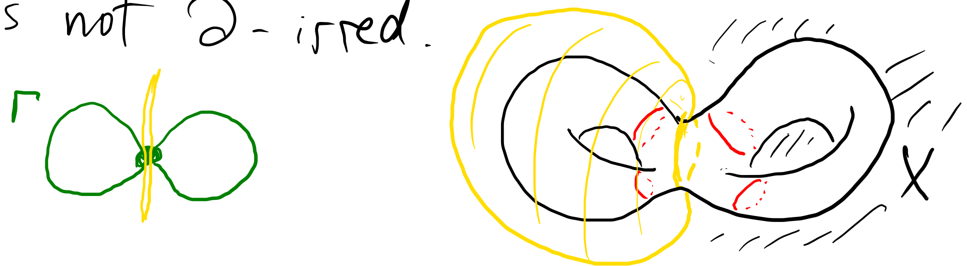


Difficulties:

- If Γ is separable, then X is reducible:

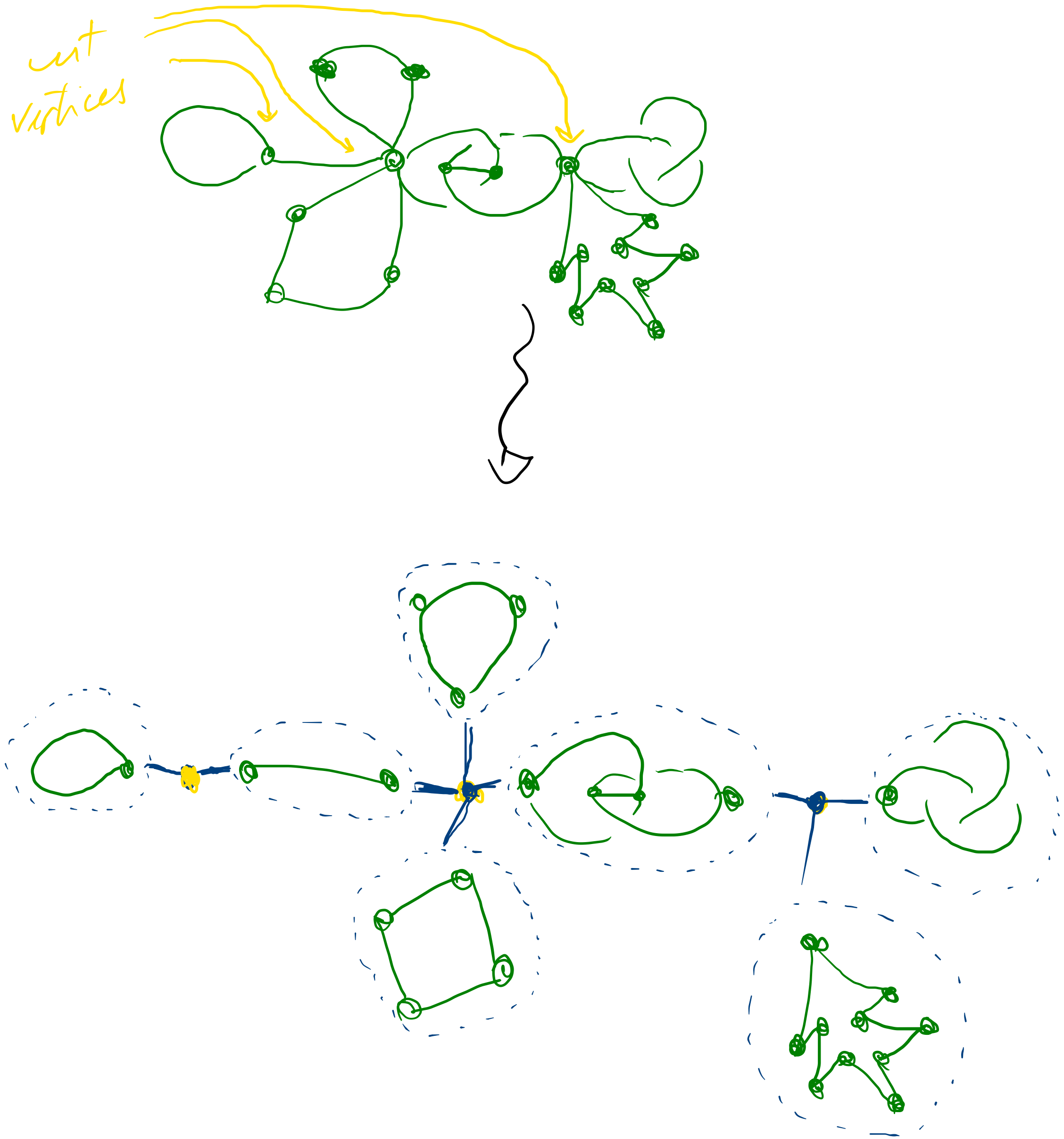


- If Γ has cut vertices, then (X, P) is not ∂-irred.



Solution:

- Decompose Γ into non-separable pieces
- Decompose non-separable pieces as unions along vertices of blocks without cut vertices ("block-cut tree")
- These decompositions are canonical! So it suffices to use Matveev on exteriors of blocks!


cut vertices