# Multiplicative constants for representations

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# Main Goal

**Goal**: Given a torsion-free lattice $\Gamma \leq G$ describe (part of) the space of representations

$$\rho : \Gamma \to H ,$$

into a locally compact group.

**Idea**: Define a number $\lambda(\rho)$ (the *multiplicative constant*) associated to $\rho$ with bounded absolute value.

**Tool**: Continuous bounded cohomology

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# Plan of the talk

1. Definition of continuous bounded cohomology
2. Measurable and essentially bounded functions
3. Examples of continuous bounded cohomology groups
4. Multiplicative constants
5. Examples of multiplicative constants

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# Continuous bounded cohomology

Let $G$ be a locally compact group.

**Definition (Continuous bounded functions)**

The space of *continuous functions* on $G^{n+1}$ is

$$C_c^n(G;\mathbb{R}) := \{f : G^{n+1} \to \mathbb{R} \mid f \text{ continuous}\} .$$

An element $f \in C_c^n(G;\mathbb{R})$ is *bounded* if

$$\|f\|_\infty := \sup_{g_0,\cdots,g_n \in G} |f(g_0,\cdots,g_n)|$$

is finite. We denote such a space by $C_{cb}^n(G;\mathbb{R})$.

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# Continuous bounded cohomology

A group element $g \in G$ acts on $f \in C^n_{c(b)}(G;\mathbb{R})$ as follows

$$(gf)(g_0, \cdots, g_n) := f(g^{-1}g_0, \cdots, g^{-1}g_n)\ .$$

**Definition ($G$-invariant functions)**

An element $f \in C^n_{c(b)}(G;\mathbb{R})$ is *$G$-invariant* if

$$gf = f$$

for every $g \in G$. We denote such a subspace as $C^n_{c(b)}(G;\mathbb{R})^G$.

# Continuous bounded cohomology

The *standard homogeneous coboundary operator* is given by

$$\delta^n : \mathrm{C}^n_{c(b)}(G;\mathbb{R}) \to \mathrm{C}^{n+1}_{c(b)}(G;\mathbb{R}) \,,$$

$$(\delta^n f)(g_0, \cdots, g_{n+1}) := \sum_{i=0}^{n+1} (-1)^i f(g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_{n+1}) \,.$$

**Definition (Continuous (bounded) cohomology)**

The *continuous (bounded) cohomology* of $G$ is given by

$$\mathrm{H}^n_{c(b)}(G;\mathbb{R}) := \mathrm{H}^n(\mathrm{C}^\bullet_{c(b)}(G;\mathbb{R})^G) \,.$$

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# The comparison map



**Definition (The comparison map)**

The natural inclusion

$$\iota^n : \mathrm{C}^n_{cb}(G;\mathbb{R}) \to \mathrm{C}^n_c(G;\mathbb{R})$$

induces a map in cohomology

$$\mathrm{comp}^n_G : \mathrm{H}^n_{cb}(G;\mathbb{R}) \to \mathrm{H}^n_c(G;\mathbb{R}) \ ,$$

called *comparison map*.

# Functoriality



Given a continuous representation $\rho : G \to H$ we have a map

$$\mathrm{C}^n_{c(b)}(\rho) : \mathrm{C}^n_{c(b)}(H; \mathbb{R}) \to \mathrm{C}^n_{c(b)}(G; \mathbb{R}) ,$$

$$\mathrm{C}^n_{c(b)}(\rho)(\psi)(g_0, \cdots , g_n) := \psi(\rho(g_0), \cdots , \rho(g_n)) .$$

Hence we have a well-defined map in cohomology

$$\mathrm{H}^n_{cb}(\rho) : \mathrm{H}^n_{cb}(H; \mathbb{R}) \to \mathrm{H}^n_{cb}(G; \mathbb{R}) .$$

**Functoriality**

Continuous cohomology and continuous bounded cohomology are *functorial*.

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### Functoriality

Continuous cohomology and continuous bounded cohomology are *functorial*.

# Measurable and essentially bounded functions

Let $(X, \nu)$ be a measure space. Suppose $G$ acts on $X$ preserving the measure class of $\nu$.

**Definition (Measurable/Essentially bounded functions)**

The space of *bounded measurable functions* on $X^{n+1}$ is

$$\mathcal{B}^\infty(X^{n+1}; \mathbb{R}) := \{f : X^{n+1} \to \mathbb{R} \mid f \text{ is bounded measurable}\} .$$

We define the space of *essentially bounded functions* on $X^{n+1}$ as

$$\mathrm{L}^\infty(X^{n+1}; \mathbb{R}) := \mathcal{B}^\infty(X^{n+1}; \mathbb{R}) / \sim ,$$

where $f \sim f'$ if they coincide almost everywhere.

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# Bounded/Essentially bounded functions

Given $f \in \mathcal{B}^\infty(X^{n+1};\mathbb{R})$ (resp. $f \in \mathrm{L}^\infty(X^{n+1};\mathbb{R})$) we define the action by $g \in G$ as

$$(gf)(\xi_0, \cdots, \xi_n) := f(g^{-1}\xi_0, \cdots, g^{-1}\xi_n) \, .$$

**Theorem ([Mon01])**

*Let $G$ be a semisimple Lie group of non-compact type and let $P \leq G$ be a minimal parabolic subgroup. Then it holds*

$$\mathrm{H}^n_{cb}(G;\mathbb{R}) \cong \mathrm{H}^n(\mathrm{L}^\infty((G/P)^{\bullet+1};\mathbb{R})^G) \, .$$

**Theorem ([BI02])**

*For any measurable $G$-space $(X,\nu)$ we have a map*

$$\mathfrak{c}^n : \mathrm{H}^n(\mathcal{B}^\infty(X^{n+1},\mathbb{R})^G) \to \mathrm{H}^n_{cb}(G;\mathbb{R}) \, .$$

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# Examples and computations



## Euler class (1)

Take $G = \text{Homeo}^+(\mathbb{S}^1)$ and $X = \mathbb{S}^1$. If we consider the orientation function

$$\mathfrak{o}(\xi_0, \xi_1, \xi_2) = \frac{1}{2} \cdot \text{the sign of the triple } \xi_0, \xi_1, \xi_2 \ ,$$

then it is a cocycle and $\mathfrak{o} \in \mathcal{B}^\infty((\mathbb{S}^1)^3; \mathbb{R})^{\text{Homeo}^+(\mathbb{S}^1)}$. The class

$$e^b_{\mathbb{R}} := \mathfrak{c}^2[\mathfrak{o}] \in \text{H}^2_b(\text{Homeo}^+(\mathbb{S}^1); \mathbb{R}) \ .$$

does not vanish and it is called *real bounded Euler class*.

## Euler class (2)

Take now $H = \mathrm{PSL}(2,\mathbb{R}) < \mathrm{Homeo}^+(\mathbb{S}^1)$ and $X = \mathbb{S}^1$. Fix $P = \mathrm{Stab}_H(\xi_0)$ for some $\xi_0 \in \mathbb{S}^1$. The group $P$ is a minimal parabolic subgroup, thus

$$\mathrm{H}^n_{cb}(\mathrm{PSL}(2,\mathbb{R});\mathbb{R}) \cong \mathrm{H}^n(\mathrm{L}^\infty((\mathbb{S}^1)^{\bullet+1});\mathbb{R})^{\mathrm{PSL}(2,\mathbb{R})} .$$

In the particular case $n = 2$, we have

$$\mathrm{H}^2_{cb}(\mathrm{PSL}(2,\mathbb{R});\mathbb{R}) \cong \mathbb{R} \cdot e^b_{\mathbb{R}} .$$

## Volume class (1)

Let $G = \mathrm{Isom}^+(\mathbb{H}^3)$ and $X = \mathbb{S}^2$. We can define

$$\mathrm{Vol}_3 : (\mathbb{S}^2)^4 \to \mathbb{R}\ , \quad \mathrm{Vol}_3(\xi_0, \cdots, \xi_3) := \int_{\Delta(\xi_0,\cdots,\xi_3)} \omega\ .$$

It a is cocycle and $\mathrm{Vol}_3 \in \mathcal{B}^\infty((\mathbb{S}^2)^4; \mathbb{R})^{\mathrm{Isom}^+(\mathbb{H}^3)}$.
We have that

$$\Theta_3^b := \mathfrak{c}^3[\mathrm{Vol}_3] \in \mathrm{H}^3_{cb}(\mathrm{Isom}^+(\mathbb{H}^3); \mathbb{R})\ .$$

does not vanish and it is called *Volume class*.

## Volume class (2)

Fix $P = \mathrm{Stab}_G(\xi_0)$ for some $\xi_0 \in \mathbb{S}^2$. The group $P$ is a minimal parabolic subgroup, thus

$$\mathrm{H}^n_{cb}(\mathrm{Isom}^+(\mathbb{H}^3);\mathbb{R}) \cong \mathrm{H}^n(\mathrm{L}^\infty((\mathbb{S}^2)^{\bullet+1}));\mathbb{R})^{\mathrm{Isom}^+(\mathbb{H}^3)} .$$

In the particular case $n=3$ the Volume class is a generator of the group, that is

$$\mathrm{H}^3_{cb}(\mathrm{Isom}^+(\mathbb{H}^3);\mathbb{R}) \cong \mathbb{R}\cdot\Theta^b_3 .$$

# Boundary maps

Let $\Gamma \leq G$ be a lattice in a semisimple Lie group of non-compact type and let $P \leq G$ be a minimal parabolic subgroup. Given a representation $\rho : \Gamma \to H$, let $(Y, \theta)$ be a $H$-measure space with the measure class of $\theta$ preserved by the $H$-action.

**Definition (Boundary map)**

A measurable map $\varphi : G/P \to Y$ is a *boundary map* if it is $\rho$-equivariant, that is

$$\varphi(\gamma\xi) = \rho(\gamma)\varphi(\xi) ,$$

for every $\gamma \in \Gamma$ and almost every $\xi \in G/P$.

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# Boundary maps

### The circle

Take $G = \mathrm{PSL}(2,\mathbb{R})$, $H = \mathrm{Homeo}^+(\mathbb{S}^1)$ and let $Y = \mathbb{S}^1$. Consider a lattice $\Gamma \leq G$ and a representation $\rho : \Gamma \to H$. If we fix $P = \mathrm{Stab}_G(\xi_0)$ for some $\xi_0 \in \mathbb{S}^1$ then $G/P \cong \mathbb{S}^1$. Thus a boundary map for $\rho$ is a measurable $\rho$-equivariant map $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ .

### The sphere

Take $G = H = \mathrm{Isom}^+(\mathbb{H}^3)$ and let $Y = \mathbb{S}^2$. Consider a lattice $\Gamma \leq G$ and a representation $\rho : \Gamma \to H$. If we fix $P = \mathrm{Stab}_G(\xi_0)$ for some $\xi_0 \in \mathbb{S}^2$ then $G/P \cong \mathbb{S}^2$. Thus a boundary map for $\rho$ is a measurable $\rho$-equivariant map $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ .

# Boundary maps



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- let $\varphi : G/P \to Y$ be a boundary map;
- let $\psi \in \mathrm{L}^\infty((G/P)^{n+1}; \mathbb{R})^G$ such that $\Psi = [\psi] \in \mathrm{H}^n_{cb}(G; \mathbb{R})$ is a **generator.**

# Multiplicative constants

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# Multiplicative constants

## Proposition ([BI09])

*For any cocycle $\psi' \in \mathcal{B}^{\infty}(Y^{n+1};\mathbb{R})^H$, there exists a real number $\lambda(\rho)$ such that*

$$\lambda(\rho)\psi(\xi_0,\cdots,\xi_n)+\text{cobound.} = \int_{\Gamma\backslash G} \psi'(\varphi(\overline{g}\xi_0),\cdots,\varphi(\overline{g}\xi_n))d\mu(\overline{g}) \,,$$

*where $\mu$ is the normalized measure on the quotient $\Gamma\backslash G$.*

## Definition (Multiplicative constant)

The real number $\lambda(\rho)$ is called *multiplicative constant* associated to $\rho$ (and to $\psi, \psi'$ to be precise).

# Multiplicative constants



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The real number $\lambda(\rho)$ is called *multiplicative constant* associated to $\rho$ (and to $\psi, \psi'$ to be precise).

# The Euler invariant

Take the following setting

- Let $G = \mathrm{PSL}(2, \mathbb{R})$ and let $\Gamma \leq G$ be a cocompact lattice (a surface group);
- Let $\Sigma := \Gamma \backslash \mathbb{H}^2$ the closed surface associated to $\Gamma$ and fix $[\Sigma] \in \mathrm{H}_2(\Sigma; \mathbb{R})$.
- Let $H = \mathrm{Homeo}^+(\mathbb{S}^1)$ and let $Y = \mathbb{S}^1$;
- Take a representation $\rho : \Gamma \to H$;

**Definition (The Euler invariant)**

The Euler invariant of $\rho$ is defined by

$$\mathrm{eu}(\rho) := \langle \mathrm{comp}^2 \circ \mathrm{H}^2_b(\rho)(e^b_{\mathbb{R}}), [\Sigma] \rangle \ .$$

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# The Euler invariant



Suppose that $\rho : \Gamma \to H$ is *not elementary*. Then it admits a boundary map $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$. Since $e^b_{\mathbb{R}}$ generates $\mathrm{H}^2_{cb}(\mathrm{PSL}(2,\mathbb{R});\mathbb{R})$ there exists a multiplicative constant $\lambda(\rho)$ such that

$$\lambda(\rho)\mathfrak{o}(\xi_0,\xi_1,\xi_2) = \int_{\Gamma\backslash \mathrm{PSL}(2,\mathbb{R})} \mathfrak{o}(\varphi(\overline{g}\xi_0),\varphi(\overline{g}\xi_1),\varphi(\overline{g}\xi_2))d\mu(\overline{g}) \ .$$

**Proposition ([Ioz02])**

*It holds* $|\lambda(\rho)| \leq 1$ *and*

$$\lambda(\rho) = \frac{\mathrm{eu}(\rho)}{\chi(\Sigma)} \ .$$

# The Euler invariant



**Theorem ([Ioz02])**

*Let $\rho : \Gamma \to H$ such that $|\mathrm{eu}(\rho)| = |\chi(\Sigma)|$. Suppose that the orbits of $\rho$ are dense $\mathbb{S}^1$, Then $\rho$ is conjugated to a hyperbolization.*

# The Volume invariant

Take the following setting

- Let $G = \mathrm{Isom}^+(\mathbb{H}^3)$ and let $\Gamma \leq G$ be a cocompact lattice;
- Let $M := \Gamma \backslash \mathbb{H}^3$ the closed hyperbolic manifold associated to $\Gamma$ and fix $[M] \in \mathrm{H}_3(M; \mathbb{R})$.
- Let $H = \mathrm{Isom}^+(\mathbb{H}^3)$ and let $Y = \mathbb{S}^2$;
- Take a representation $\rho : \Gamma \to H$;

**Definition (The Volume invariant)**

The Volume invariant of $\rho$ is defined by

$$\mathrm{Vol}(\rho) := \langle \mathrm{comp}^3 \circ \mathrm{H}^3_b(\rho)(\Theta^b_3), [M] \rangle \, .$$

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# The Volume invariant

Suppose that $\rho : \Gamma \to H$ is *not elementary*. Then it admits a boundary map $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$. Since $\Theta_3^b$ generates $\mathrm{H}^3_{cb}(\mathrm{Isom}^+(\mathbb{H}^3);\mathbb{R})$ there exists a multiplicative constant $\lambda(\rho)$ such that

$$\lambda(\rho)\mathrm{Vol}_3(\xi_0,\ldots,\xi_3) = \int_{\Gamma\backslash \mathrm{Isom}^+(\mathbb{H}^3)} \mathrm{Vol}_3(\varphi(\overline{g}\xi_0),\cdots,\varphi(\overline{g}\xi_3))d\mu(\overline{g}) \; .$$

**Proposition ([BBI13])**

*It holds* $|\lambda(\rho)| \leq 1$ *and*

$$\lambda(\rho) = \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \; .$$

# The Euler invariant



**Theorem ([BBI13])**

*Let $\rho : \Gamma \to H$ such that $|\mathrm{Vol}(\rho)| = \mathrm{Vol}(M)$. Then $\rho$ is conjugated to the standard inclusion $\Gamma \to \mathrm{Isom}^+(\mathbb{H}^3)$.*

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Thanks for your attention.