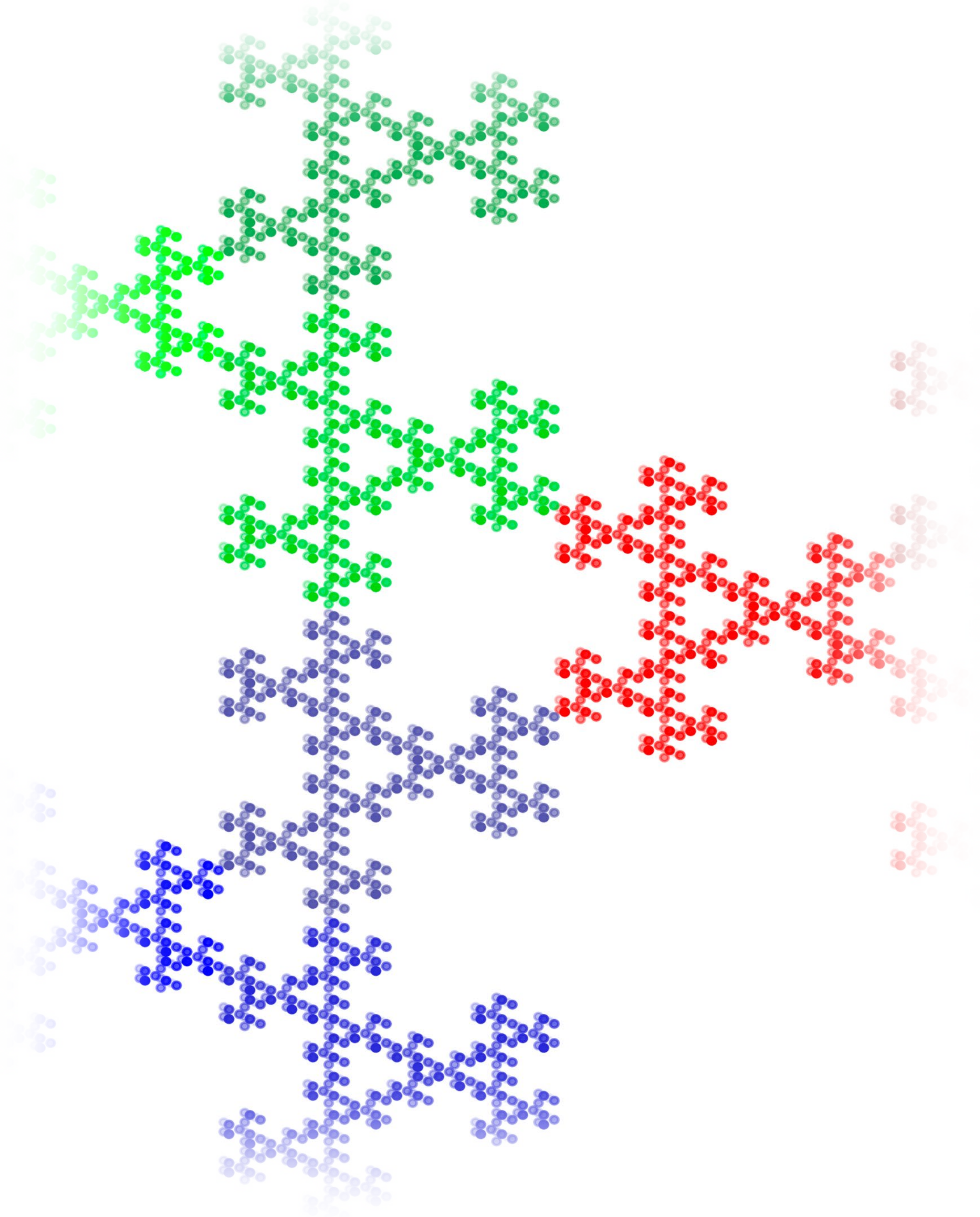

DËMUSHKIN GROUPS AND L^2 INVARIANTS

pro-p analogues of surface groups

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PRO-P SURFACE GROUPS?

- G is pro-p if it is the inverse limit of finite p-groups.
  - Compact, Hausdorff and totally disconnected.
- Examples: finite p-groups,
$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^k\mathbb{Z}\,.$$
- Pro-p completions: take an *abstract* group Γ and set
$$\hat{\Gamma}_p = \varprojlim \Gamma/N\,,$$
where $(\Gamma : N) = p^k$.
- Fin. gen. free groups $\xrightarrow{\hat{\Gamma}_p}$ fin. gen. free *pro-p* groups.
- Orientable surface groups $\xrightarrow{\hat{\Gamma}_p}$ examples of…

DËMUSHKIN GROUPS!

- Pro-p analogues of "abstract" surface groups.
- *Def.:* satisfy the pro-p Poincaré duality in dim. 2:

$$\cup\colon H_c^i(G,\mathbb{F}_p)\times H_c^{2-i}(G,\mathbb{F}_p)\to H_c^2(G,\mathbb{F}_p)\simeq \mathbb{F}_p$$

is nondegenerate.

- [Dëmushkin, 1961] For odd p, we have even d and:

$$G\simeq \left\langle x_1,x_2,\cdots,x_{d-1},x_d \middle| x_1^q[x_1,x_2]\cdots[x_{d-1},x_d]=1\right\rangle_p.$$

Сергей Петрович Дёмушкин

HOMOLOGY OF PRO-P GROUPS

- Completed group algebra:

$$[\![\mathbb{F}_p G]\!] = \varprojlim [\mathbb{F}_p(G/U)].$$

- Homology as $\mathrm{Tor}_i^{[\![\mathbb{F}_p G]\!]}(\mathbb{F}_p, M) = H_i(G, M)$.

- $H_i(G, \mathbb{F}_p)$ is the dual $\mathbb{F}_p$-vector space of $H_c^i(G, \mathbb{F}_p)$.

- $\dim_{\mathbb{F}_p} H_1(G, \mathbb{F}_p)$ = minimal # top. generators, $\dim_{\mathbb{F}_p} H_2(G, \mathbb{F}_p)$ = minimal # of relators.

L^2 INVARIANTS

- Classical L^2-Betti numbers: passing to a Γ-invariant setting using the universal cover.
- For nice fields K, abstract groups Γ and residual chains $\{\Lambda_i\}$, they satisfy Lück approximation:

$$b_n^{(2)}(\Gamma; K) = \lim_{i\to\infty} \frac{b_n(\Lambda_i; K)}{(\Gamma:\Lambda_i)}.$$

- For pro-p groups G and $[\![\mathbb{F}_p G]\!]$-modules M, we *define* them through Lück approximation:

$$b_n^{(2)}(G; M) = \inf_{U \trianglelefteq_o G} \frac{\dim_{\mathbb{F}_p} H_n(U,M)}{(G:U)}.$$

- $b_1^{(2)}(G, \mathbb{F}_p)$ is the rank gradient of G.

WHAT DO WE GET FOR DËMUSHKIN GROUPS*?

[Jaikin-Zapirain, Shusterman 2019]:

- The *Atiyah Conjecture*: when defined, $b_n^{(2)}(G,M)$ is an integer!
- *Kaplansky's Conjecture*: $[\![\mathbb{F}_p G]\!]$ has no zero divisors.
- The *Strenghtened Hanna Neumann Inequality* for large G and $H,K \leq_c G$ fin. gen.:

$$\sum_{x\in H\backslash G/K} \overline{\mathrm{rk}}(H\cap xKx^{-1}) \leq \overline{\mathrm{rk}}(H)\cdot\overline{\mathrm{rk}}(K)\,,$$

  where $\overline{\mathrm{rk}}(H) = \max(\mathrm{rk}(H)-1, 0)$ is the reduced rank.

+ [Antolín, Jaikin-Zapirain 2020] + [S.]:

- Retracts of Dëmushkin groups are inert, that is, if $G \simeq N \rtimes H$, then, for every $K \leq_c G$, we have:

$$\mathrm{rk}(H\cap K) \leq \mathrm{rk}(K)$$

*for infinite Dëmushkin groups, which means all of them except $\mathbb{Z}/2\mathbb{Z}$.