L^2-Betti numbers and Computability of reals

Matthias Uschold

9.5.22

L^2-Betti numbers and Computability of reals

1. Intro L^2-Betti numbers
2. Recent results by Löh-U
3. Why should you care about L^2-Betti numbers?
   → Conjecture by Gromov

$\boxed{1}$ INTRO L^2-BETTI NUMBERS

<u>Setup:</u> G: countable group

X: CW-cplx w/ free, <u>finite type</u>, cellular action $G \curvearrowright X$

i.e. X_n/G is compact

<u>Recall:</u> $b_n(X,\mathbb{C}) := \dim_{\mathbb{C}} H_n(X,\mathbb{C}) \in \mathbb{N} \cup \{\infty\}$

$\boxed{1}$ INTRO L^2-BETTI NUMBERS

Setup: G: countable group

X: CW-cplx w/ free, finite type, cellular action $G \curvearrowright X$

i.e. X_n/G is compact

Recall: $b_n(X, \mathbb{C}) := \dim_{\mathbb{C}} H_n(X, \mathbb{C}) \in \mathbb{N}$

Def: $b_n^{(2)}(G \curvearrowright X) := \dim_G H_n^{(2)}(G \curvearrowright X) \in \mathbb{R}_{\geq 0}$

$H_n^{(2)}(G \curvearrowright X) =$ reduced homology of $C_*^{\text{cell}}(X) \otimes_{\mathbb{Z}G} \ell^2 G$

$$\ell^2 G = \{ a : G \to \mathbb{C} \mid \sum_{g \in G} |a_g|^2 < \infty \}$$

Def: $b_n^{(2)}(G \curvearrowright X) := \dim_G H_n^{(2)}(G \curvearrowright X) \in \mathbb{R}_{\geq 0}$

$\dim_G$ = von Neumann dimension.

1. in gen $\in \mathbb{R}_{\geq 0}$ (not $\in \mathbb{N}$)
2. $\dim_G A = 0 \iff A \cong 0$ (faithfulness)
3. $\dim_G(\ell^2 G) = 1$ (normality)
4. additive under short (weakly) exact sequences
5. If $H \subset G$ of finite index, then

$$\dim_H A = [G:H] \cdot \dim_G A$$

by restriction.

L^2-Betti numbers share many properties of (ordinary) Betti numbers.

Additional power: Restriction property:

If $H \subset G$ fin index subgroup, then

$$b_n^{(2)}(H \curvearrowright X) = [G:H]\, b_n^{(2)}(G \curvearrowright X)$$

L^2-Betti numbers share many properties of (ordinary) Betti numbers.

Additional power: Restriction property:

If $H \subset G$ fin index subgroup, then

$$b_n^{(2)}(H \curvearrowright X) = [G:H]\, b_n^{(2)}(G \curvearrowright X)$$

<u>Example:</u> (G finite)

$$b_n(X, \mathbb{C}) = b_n^{(2)}(1 \curvearrowright X) = |G| \cdot b_n^{(2)}(G \curvearrowright X)$$

$$\Rightarrow b_n^{(2)}(G \curvearrowright X) = \frac{1}{|G|} \cdot b_n(X, \mathbb{C})$$

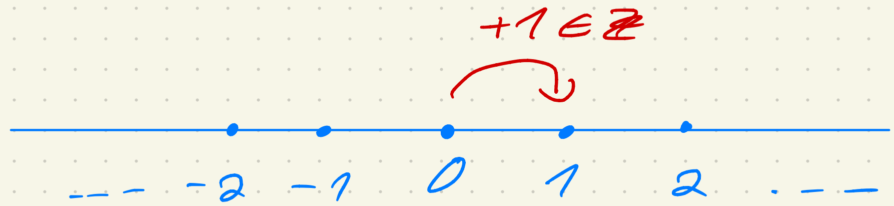
Ex (degree 0)

If X is connected,

$$b_0^{(2)}(G \curvearrowright X) = \frac{1}{|G|}$$

$$\left(\frac{1}{\infty} := 0 \right)$$

Ex $\mathbb{Z} \curvearrowright \mathbb{R}$



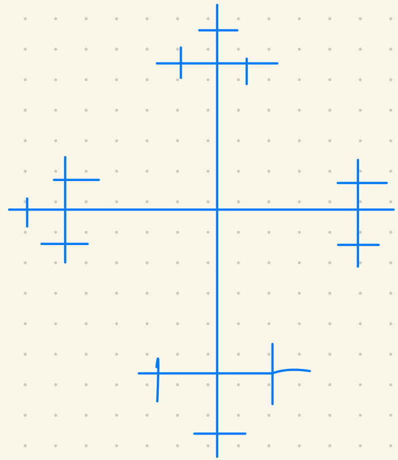
$$\forall_{n\in\mathbb{N}} \quad b_n^{(2)}(\mathbb{Z} \curvearrowright \mathbb{R}) = 0$$

Later: not a surprise: $\mathbb{R} = E\mathbb{Z} = \widetilde{B\mathbb{Z}}$ and $\mathbb{Z}$ is amenable

Cheeger-Gromov $\Rightarrow$ all ℓ^2-Betti numbers vanish.

Ex:

$$F_2 \curvearrowright \widetilde{S_1 \vee S_1} =: EF_2$$



$$b_n^{(2)}(F_2 \curvearrowright EF_2) = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

Modern viewpoint: homological L^2-Betti numbers

classical viewpoint: cohomological

[Cheeger-Gromov]

finite type $\Rightarrow$ $b_n^{(2)} = b^n_{(2)}$

$\boxed{2}$ RECENT RESULTS

Q (Atiyah '76): Can $b_n^{(2)}(G \curvearrowright X)$ be irrational?

A (Austin '13): Yes. Even: uncountably many numbers can occur.

A (Grabowski '14, Pichot-Schick-Żuk '15): All numbers in $\mathbb{R}_{\geq 0}$ can occur.

Q (refined): Which numbers occur? (given some hypotheses on G)

Def: The L^2-Betti numbers arising from G are all numbers of the form

$$b_n^{(2)}(G \curvearrowright X)$$

for some $n \in \mathbb{N}$

X: free, fin type G-CW-cplx.

Def: The L^2-Betti numbers arising from G are all numbers of the form

$$b_n^{(2)}(G \curvearrowright X)$$

for some $n \in \mathbb{N}$,

X: free, fin type G-CW-cplx.

Def: The L^2-Betti numbers of G are all numbers of the form

$$b_n^{(2)}(G \curvearrowright EG)$$

for some $n \in \mathbb{N}$

$EG = \widetilde{BG}$: classifying space

Thm (Groth '12, Pichot-Schick-Żuk '15, Löh-U '22)

The set of L^2-Betti numbers arising from all finitely generated groups

- with solvable word problem, and
- sofic (e.g. free or amenable)

is equal to $EC_{\geq 0}$.

set of non-neg real numbers with computable binary expansion (effectively computable numbers)

Def: $r \in \mathbb{R}$ is effectively computable if there ex. a Turing machine (an algorithm, a computer program) that outputs the binary expansion of r.
(equivalently: decimal)

Ex: algebraic numbers, π, e

$3.14159 \ldots\ldots$

Thm (Groth '12, Pichot-Schick-Żuk '15, Löh-U '22)

The set of L^2-Betti numbers arising from all finitely generated groups

- with solvable word problem, and
- sofic (e.g. free or amenable)

is equal to $EC_{\geq 0}$.

Proof of "is contained in", i.e. $b_n^{(2)}(G \curvearrowright X) \in EC_{\geq 0}$

$A \in M_{n \times n}(\mathbb{Z}G) \rightarrow \mathrm{tr}_G (1-A)^k$

need entries on diag corr to $e \in G$.

"Easier case": G finitely presented, residually finite.

<u>Thm</u> (Lück approximation)
$(G_i)_{i\in\mathbb{N}}$ be a residual chain of G.

$$b_n^{(2)}(G \curvearrowright X) = \lim_{i\to\infty} \frac{b_n(G_i\backslash X, \mathbb{C})}{[G:G_i]}$$

<u>Strategy</u>: ① Need to find a res. chain $(G_i)_{i\in\mathbb{N}}$.
s.t. $i \longmapsto \frac{b_n(G_i\backslash X, \mathbb{Q})}{\underbrace{[G:G_i]}_{\in\mathbb{Q}}}$ is algorithmically computable.

② Bound the "error" / "speed of convergence"

③ Need result on EC numbers: $\in EC$

In general (G fin gen, sofic, solvable word problem):

Strategy:

① Describe L^2-Betti numbers using a measure-theoretic settings (⇝ spectral measures)

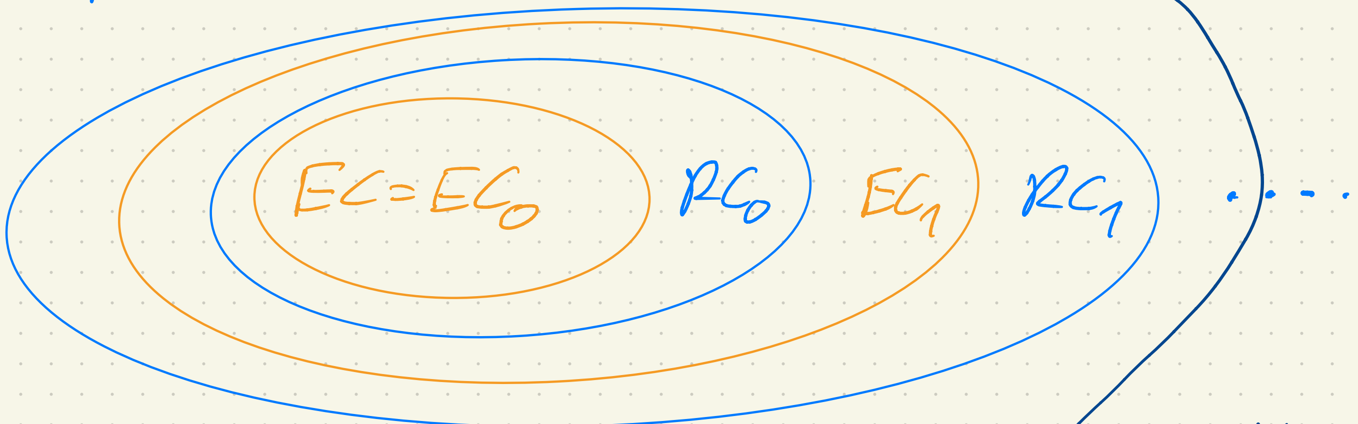
② Find some approximation that is computable

③ Bound error

⇝ algo that computes binary expansion of $b_n^{(2)}(G \curvearrowright X)$

<u>Thm:</u> (Löh–U '22) All L^2-Betti numbers arising from finitely presented groups are contained in RC_1

right-computable numbers of Turing degree 1



<u>Know:</u> $EC_{\geq 0} \subseteq \underbrace{\{L^2\text{-B.N. ar. from fin pres.}\}}_{=?} \subseteq (RC_1)_{\geq 0}$

Key: word problem in fin pres. groups is semi-decidable.

3 WHY SHOULD YOU CARE ABOUT L^2-BETTI NUMBERS?

<u>Conjecture</u> (Gromov) M: asph., occ mfd.

If $\|M\| = 0$, then, for all $n \in \mathbb{N}$

$$b_n^{(2)}(M) := \underbrace{b_n^{(2)}(\pi_1(M) \curvearrowright \widetilde{M})}_{= b_n^{(2)}(\pi_1(M))} = 0$$

In particular:

$$\chi(M) = \sum_{n \in \mathbb{N}} (-1)^n \cdot b_n^{(2)}(M) = 0$$

Common properties

- multiplicative under finite coverings / subgroup
- satisfy proportionality principle with volume.
- vanishing results in the amenable case:

Thm (Cheeger–Gromov '86)
G: countable, amenable group.
Then, for $n \geq 1$:

$$b_n^{(2)}(G) = 0$$

$$\parallel$$

$$b_n^{(2)}(G \curvearrowright EG)$$

Thm (Gromov)
M: asph., occ mfd,
$\pi_1(M) \neq 1$ amenable, then
$\|M\| = 0$.

(Generalisation: If exists an amenable covers of mult $\leq$ dim)

Known result:

Thm: M^d. aspherical, occ mfd

$$\sum_{n=0}^{d} b_n^{(2)}(M) \leq (d+1) \underbrace{|M|}_{\text{Integral foliated simp. vol.}}$$

Corollary: If $|M| = 0$, then $b_n^{(2)}(M) = 0$

THANK YOU!

<u>Ex</u>: (a real number $\notin EC$)

Let $H \subset \mathbb{N}$ be a subset that is not decidable (e.g. the Halting set, see below). Then,

$$\sum_{n \in H} 2^{-n} \notin EC$$

<u>Ex</u>: (Halting set)

Let $M_0, M_1, \ldots$ be an (algorithmically enumerable) list of all possible Turing machines. Then, the <u>Halting set</u> is

$$H := \{ n \in \mathbb{N} \mid M_n \text{ halts on empty input} \} \subset \mathbb{N}$$