Volumes and random walks on mapping class groups

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June 22, 2020

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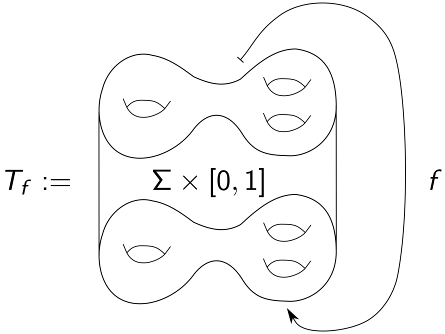

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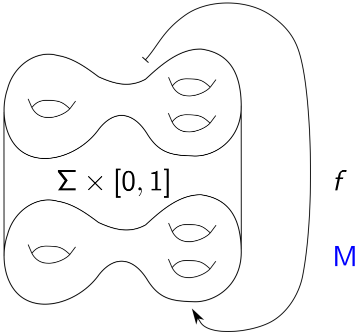

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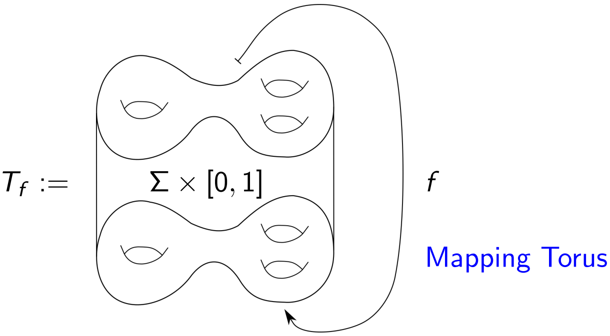

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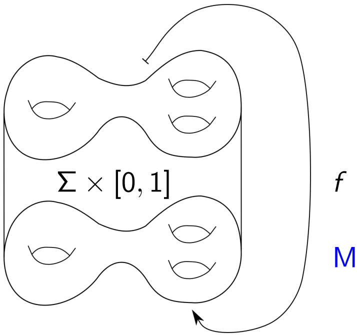

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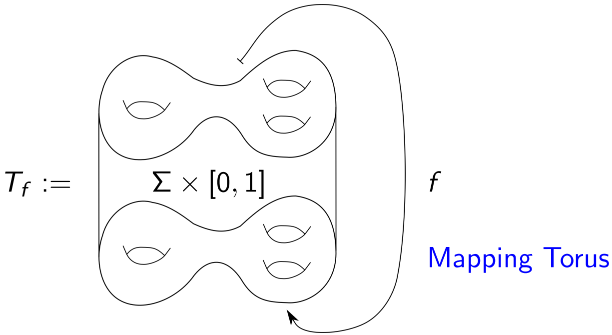

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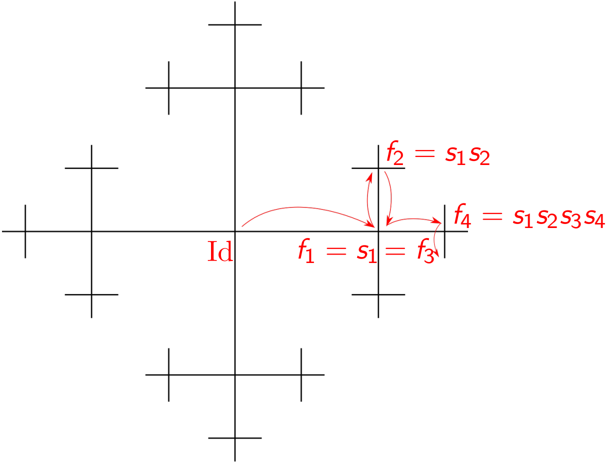

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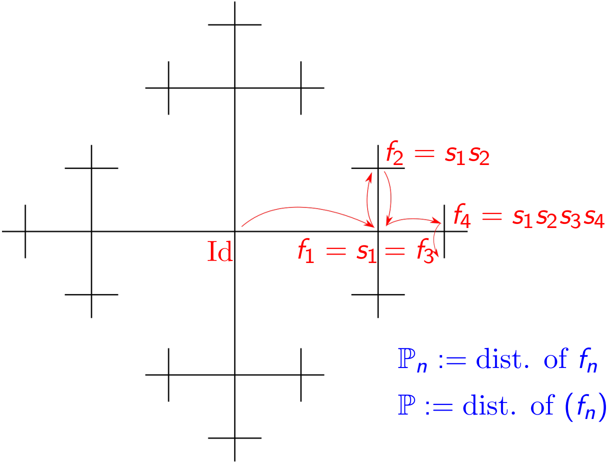

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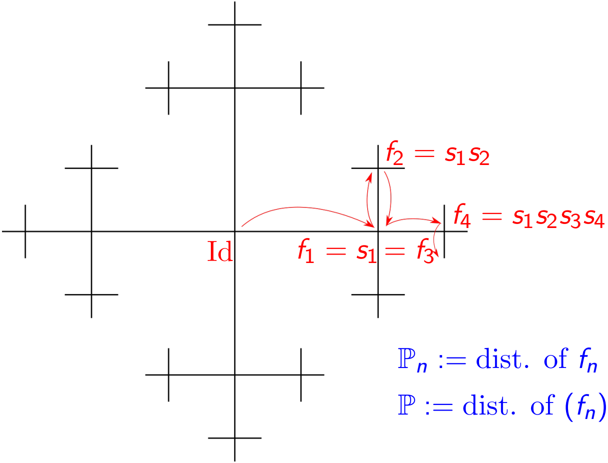

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Family of random mapping tori $:= (T_{f_n})_{\in\mathbb{N}}$ where $(f_n)_{n\in\mathbb{N}}$ is a random walk on $\mathrm{Mod}(\Sigma)$ driven by the uniform probability on S.

A law of large numbers

We have the following *law of large numbers* for the volume

Theorem A (V. 2019)

There exists $v = v(S) > 0$ *such that for* $\mathbb{P}$*-almost every* $(f_n)_{n\in\mathbb{N}}$

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  Say $f = s_1 \cdots s_n$. Extend $\tau \times \{0\} \sqcup (f\tau) \times \{n\}$ to a triang. of $\Sigma \times [0,n]$ with $\leq Kn$ simpl. by stacking triang. of $\Sigma \times [0,1]$.

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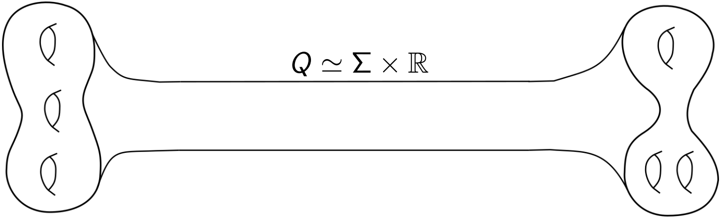
- Same holds for families $M_{f_n} := H_g \cup_{f_n} H_g$ of *random Heegaard splittings* with exactly the *same asymptotic value* $v = v(S)$.

Quasi-fuchsian manifolds

Quasi-fuchsian manifold:



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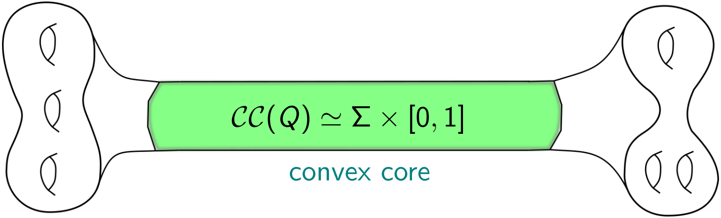

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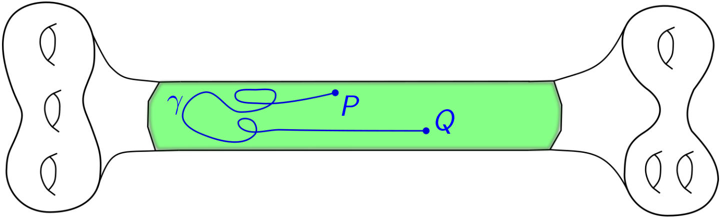

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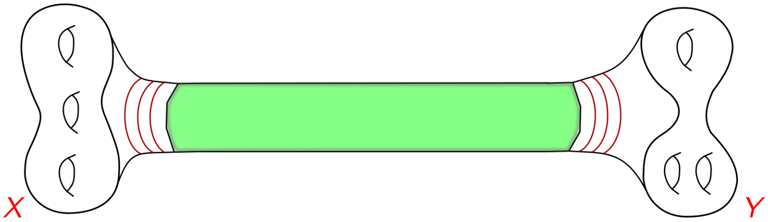

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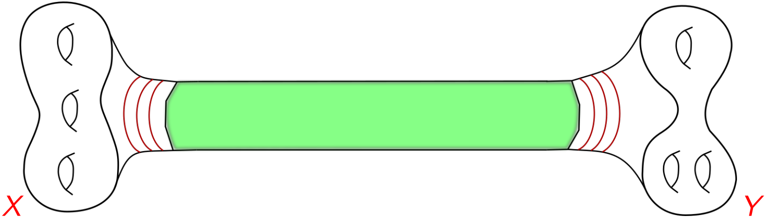

Bers: For every $(X, Y) \in \mathcal{T} \times \mathcal{T}$ there is a unique $Q = Q(X, Y)$.

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Theorem B (V. 2019)

Fix $o \in \mathcal{T}$. *For* $\mathbb{P}$*-almost every* $(f_n)_{n\in\mathbb{N}}$

$$\lim_{n\to\infty} \frac{\mathrm{vol}(\mathcal{CC}(Q(o, f_n o)))}{n} = v_3 \cdot v(S),$$

where $v(S) =$ *same as before and* $v_3 =$ *vol. reg. ideal tetrahedron.*

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- Estimates on $d\mathrm{vol} : T(\mathcal{T} \times \mathcal{T}) \to \mathbb{R}$ with respect to $|| \bullet ||_{\mathrm{WP}}$ and $|| \bullet ||_{\mathcal{T}}$ [Schlenker, Kojima-McShane]

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Thus Theorem B $\Longrightarrow$ Theorem A.

The proof of the Proposition

Besson, Courtois and Gallot: If (T_f, ρ) is a metric that satisfies $\sec \in (-1-\epsilon, -1+\epsilon)$ for some $0 < \epsilon < 1/2$, then

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for some universal function $o(\epsilon)$.

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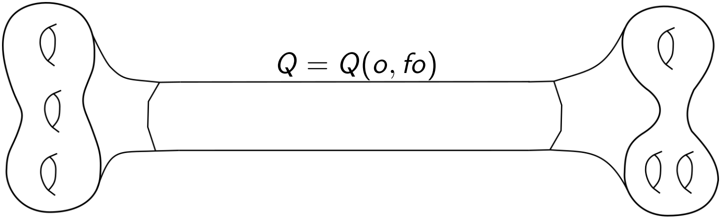

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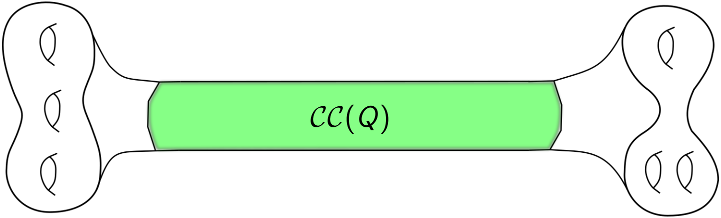

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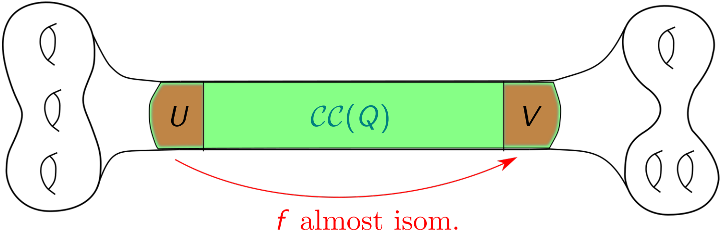

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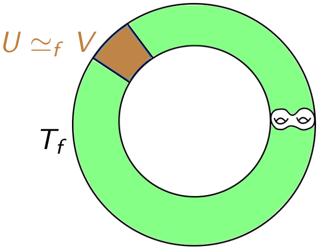

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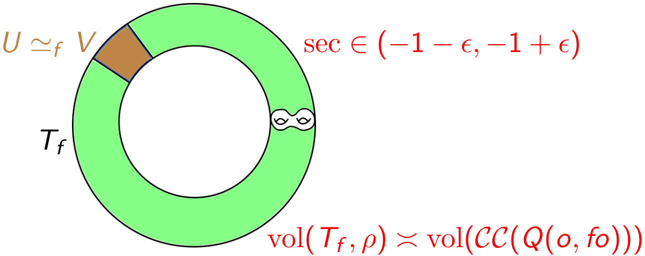

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Remark.

Inflexibility $\Rightarrow |n \cdot \mathrm{vol}(T_f, \sigma) - \mathrm{vol}(\mathcal{CC}(Q(o, f^n o)))| = O(1)$
[Kojima-McShane, Brock-Bromberg]

The proof of Theorem B

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$$f_n = (s_1 \cdots s_N) \dots (s_{N(m-1)+1} \cdots s_{Nm}) = h_1 \cdots h_m.$$

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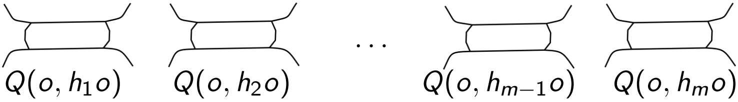

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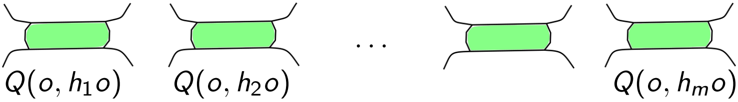

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