Critical problems in Carnot groups

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The problems

This presentation is about two works developed jointly with Biagi-Vecchi and Vecchi. The focus is on the equation

$$\begin{cases} -\Delta u = \lambda u^s + u^{2^*-1} & \text{in } \Omega \Subset \mathbb{R}^N \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathsf{P}_\lambda^{(s)}$}$$

- Ω is bounded, connected and with smooth enough boundary
- $s \in (-1, 0)$ (*singular case*) or $s \in (0, 1)$ (*concave-convex case*), $\lambda > 0$
- $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent

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$$\begin{cases} -\Delta_{\mathbb{G}} u = \lambda u^s + u^{2^*_Q - 1} & \text{in } \Omega \Subset \mathbb{G} \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathrm{P}_\lambda^{(s)}$}$$

- Ω is bounded, connected and with smooth enough boundary
- $s \in (-1, 0)$ (*singular case*) or $s \in (0, 1)$ (*concave-convex case*), $\lambda > 0$
- $2^*_Q = \frac{2Q}{Q-2}$ is the critical Sobolev exponent
- $\mathbb{G}$ is a Carnot group of homogeneous dimension Q
- $\Delta_{\mathbb{G}}$ is the sub-Laplacian on $\mathbb{G}$

What is a Carnot group?

A Carnot group $\mathbb{G} = (\mathbb{R}^N, \diamond)$ of step k is a connected and simply connected Lie group such that $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})$ admits a stratification of step k,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k, \quad [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \quad \mathfrak{g}_i = \{0\} \text{ iff } i > k.$$

We denote by $X_1, \dots X_N$ a basis of left-invariant vector fields such that

- $\mathfrak{g}_i = \langle X_{m_{i-1}+1}, X_{m_{i-1}+2}, \dots, X_{m_i} \rangle$ for all $i = 1, \dots, k$ with $m_0 = 0$,
- there is a scalar product $\langle , \rangle$ on $\mathfrak{g}_1$ and $X_1, \dots, X_{m_1}$ are orthonormal,
- $n_i := m_i - m_{i-1} = \dim \mathfrak{g}_i$.

The homogeneous dimension of $\mathbb{G}$ is

$$Q := \sum_{i=1}^{k} i \cdot n_i.$$

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Given a (smooth enough) function $u \colon \mathbb{G} \to \mathbb{R}$, its horizontal gradient is

$$\nabla_{\mathbb{G}} u := (X_1 u, \dots, X_{m_1} u).$$

Given a horizontal vector field $V = v_1 X_1 + \cdots + v_{m_1} X_{m_1}$, its horizontal divergence is,

$$\mathrm{div}_{\mathbb{G}} V = X_1 v_1 + \cdots X_{m_1} v_{m_1},$$

both are independent of the choice of basis.

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both are independent of the choice of basis.

The sub-Laplacian is the differential operator

$$\Delta_{\mathbb{G}} u := \mathrm{div}_{\mathbb{G}}(\nabla_{\mathbb{G}} u) = X_1^2 u + \cdots + X_{m_1}^2 u.$$

The sub-Laplacian is clearly not elliptic, but it is *hypoelliptic*.

How a Carnot group behave?

A Carnot group possesses a family of anisotropic dilations

$$\delta_\lambda(g_1, \dots, g_N) := (\lambda^{w_1} g_1, \dots \lambda^{w_N} g_N) \quad \forall \lambda > 0,$$

where $w_j = i$ if $m_{i-1} < j \leq m_i$.

- $\nabla_{\mathbb{G}}(u \circ \delta_\lambda) = \lambda(\nabla_{\mathbb{G}} u) \circ \delta_\lambda$ and $\Delta_{\mathbb{G}}(u \circ \delta_\lambda) = \lambda^2(\Delta_{\mathbb{G}} u) \circ \delta_\lambda$,
- the Lebesgue measure $\mathcal{L}^N$ is the Haar measure on $\mathbb{G}$ and $\delta_\lambda^* \mathcal{L}^N = \lambda^Q \mathcal{L}$.

Functional setting on Carnot groups

A subRiemannian Sobolev inequality is verified on any open subset $\Omega \subset \mathbb{G}$ of a Carnot group,

$$\|f\|_{2^*_Q} \leq C \, \||\nabla_{\mathbb{G}} f|\|_2 \quad \forall f \in C_0^\infty(\Omega). \qquad (\star)$$

Here, $2^*_Q = \frac{2Q}{Q-2}$ is the sub-elliptic critical Sobolev exponent.

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We define the Folland-Stein space $S_0^1(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|f\|_{S_0^1(\Omega)} := \||\nabla_{\mathbb{G}} f|\|_2$. Observe that

$$S_0^1(\Omega) = \left\{u \in L^{2^*_Q}(\Omega) : \; X_i u \in L^2(\Omega), \; \forall 1 \leq i \leq m_1\right\}.$$

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The Folland-Stein space has a Hilbert space structure with

$$\langle u, v \rangle_{S_0^1(\Omega)} = \int_\Omega \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} v \rangle_{\mathfrak{g}_1}.$$

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The best Sobolev constant $S_{\mathbb{G}}$ in $(\star)$ is attained by a function $T \in S_0^1(\mathbb{G})$ (see [GV01]) which is also a solution (up to a constant) to

$$-\Delta_{\mathbb{G}} u = u^{2^*_Q - 1} \; \text{ in } \mathbb{G}.$$

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We introduce a rescaled function...

$$T_\varepsilon(g) := \frac{1}{\varepsilon^{(Q-2)/2}} T(\delta_{1/\varepsilon}(g))$$

$$\||\nabla_{\mathbb{G}} T_\varepsilon|\|_2^2 = \|T_\varepsilon\|_{2^*_Q}^{2^*_Q} = S_{\mathbb{G}}^{Q/2}$$

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Here, $2^*_Q = \frac{2Q}{Q-2}$ is the sub-elliptic critical Sobolev exponent.

We define the Folland-Stein space $S^1_0(\Omega)$ as the completion of $C^\infty_0(\Omega)$ with respect to the norm $\|f\|_{S^1_0(\Omega)} := \||\nabla_{\mathbb{G}} f|\|_2$. Observe that

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...and its truncated version

$$U_\varepsilon(g) := \varphi(g) T_\varepsilon(g), \quad \varphi \text{ cut-off function}$$

$$\||\nabla_{\mathbb{G}} U_\varepsilon|\|_2^2 = S_{\mathbb{G}}^{Q/2} + o\left(\varepsilon^{Q-2}\right)$$
$$\|U_\varepsilon\|_{2^*_Q}^{2^*_Q} = S_{\mathbb{G}}^{Q/2} + o\left(\varepsilon^{Q}\right).$$

The main results

$$\begin{cases} \Delta_{\mathbb{G}} u = \lambda u^s + u^{2^*_Q - 1} & \text{in } \Omega \Subset \mathbb{G} \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathsf{P}_\lambda^{(s)})$$

We consider the problem for some fixed $\lambda > 0$, and we prove – both in the singular and concave-convex case – a behaviour analogous to what happens in the Euclidean case. In particular, there exists $\Lambda > 0$ real number such that,

1. the problem admits two weak solutions for $\lambda \in (0, \Lambda)$,
2. the problem admits at least one weak solution for $\lambda = \Lambda$,
3. the problem does not admit any weak solution for $\lambda \in (\Lambda, +\infty)$.

First solution

The variational method

To prove the existence of *at least one* solution, we use an adaptation of Struwe variational method [Str08].

The main tool is the following result, where various elements introduced will be made precise in the next slides

Theorem

Consider S a reflexive Banach space and $M \subset S$ weakly closed. If $I : M \to \mathbb{R}$ respects

- $I(u) \to +\infty$ *as* $\|u\| \to +\infty$ *in* M,
- I *is weakly lower semi-continuous, i.e. for any weakly converging sequence* $u_n \rightharpoonup u$, $u_n, u \in M$, $I(u) \leq \liminf I(u_n)$,

then I is bounded from below on M and attains $\inf_M I$.

The variational method

To prove the existence of *at least one* solution, we use an adaptation of Struwe variational method [Str08].

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In our case,

- $S = S_0^1(\Omega)$ is the opportune Sobolev space to define the weak solutions
- $M = \{u \in S_0^1(\Omega) : \ \underline{u} \le u \le \overline{u} \text{ a.e.}\}$, where $\underline{u}, \overline{u}$ are a sub-solution and a super-solution to the same problem
- $I = I_\lambda$ is the functional associated to the equation $(\mathrm{P}_\lambda^{(s)})$

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Weak (sub/super)-solutions

We say that a function $u \in S_0^1(\Omega)$ is a weak sub(or super)-solution to the equation, if $u > 0$ (in the singular case $s \in (-1, 0)$ we also ask $u \in L^1_{\mathrm{loc}}(\Omega)$) and

$$\forall \varphi \in C_0^\infty(\Omega) \text{ such that } \varphi \geq 0,$$
$$\int_\Omega \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1} \leq \text{(or } \geq \text{)}\ \lambda \int_\Omega u^s \varphi + \int_\Omega u^{2^*_Q - 1} \varphi.$$

The I_λ functional

The functional associated to the problem $(\mathsf{P}_\lambda^{(s)})$ is

$$I_\lambda(u) := \frac{1}{2}\int_\Omega |\nabla_{\mathbb{G}} u|^2 - \frac{\lambda}{s+1}\int_\Omega u^{s+1} - \frac{1}{2_Q^*}\int_\Omega u^{2_Q^*}.$$

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The functional is coercive and w.l.s.c. on $M = \{\underline{u} \leq u \leq \overline{u}\}$, therefore if we show that we can build M non-empty, we automatically have the existence of a minimum.

Existence of sub-solutions

To find a sub-solution we consider the problem

$$\begin{cases} \Delta_{\mathbb{G}} u = \lambda u^s & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem has a unique weak solution in $w_\lambda \in S_0^1(\Omega) \cap L^\infty(\Omega)$, which is also a global minimizer in $S_0^1(\Omega)$ of the associated functional $J_\lambda(w_\lambda)$, and moreover $J_\lambda(w_\lambda) < 0$.

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- The concave-convex case $s \in (0, 1)$ has been proven in [BPV22] with Brezis-Oswald-type techniques,
- for the singular case $s \in (-1, 0)$, we adapt a technique of [YSY01], exploiting a classic Stampacchia scheme.

The Λ threshold

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In the singular case, $I_\lambda(tv) < 0$ for t sufficiently small, therefore we can find a minimizer for I_λ on a small ball $B_r(0)$ when λ is sufficiently small, and this minimizer is a solution.

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In the concave-convex case, by taking the unique solution to $-\Delta_{\mathbb{G}} V = 1$, and $u_1 := m_\lambda V$, for λ sufficiently small we have

$$-\Delta_{\mathbb{G}} u_1 = m_\lambda u_1 > \lambda u_1^s + u_1^{2_Q^*-1},$$

and we conclude with a sub/super-solution method.

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- $\Lambda > 0$
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Consider as a test the first eigenfunction e_1 of $-\Delta_{\mathbb{G}}$, then

$$\mu_1 \int_\Omega u e_1 = \int_\Omega (\lambda u^s + u^{2^*_Q - 1}) e_1,$$

but it always exists $\overline{\Lambda}$ such that

$$\overline{\Lambda} x^s + x^{2^*_Q - 1} > \mu_1 x \ \forall x \in \mathbb{R}^+,$$

thus setting a bound above for Λ.

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$$\Lambda := \sup\left\{\lambda > 0 : \text{ the problem } (\mathsf{P}_\lambda^{(s)}) \text{ has a weak solution}\right\}.$$

- $\Lambda > 0$
- $\Lambda < +\infty$

We use a sub/super-solution method for every $\lambda \in (0, \Lambda)$. The sub-solution $\underline{u}_\lambda$ was presented before, the super-solution is obtained by observing that it always exists $\lambda' \in (\lambda, \Lambda)$ such that $u_{\lambda'}$ is a solution of $(\mathsf{P}_{\lambda'}^{(s)})$. Therefore $u_{\lambda'}$ is a super-solution of $(\mathsf{P}_\lambda^{(s)})$.

A minimum of I_λ is a weak solution

We apply Struwe's scheme to $M := \{u \in S_0^1(\Omega) : \ \underline{u}_\lambda \leq u \leq u_{\lambda'}\}$ and we denote by u_λ the minimum of I_λ on M.

By adapting [Hai03] approach, we get that u_λ is a weak solution for ($\mathrm{P}_\lambda^{(s)}$).

Moreover, by building a bounded sequence u_{λ_k} with $\lambda_k \to \Lambda$, we can retrieve also a weak solution u_Λ for ($\mathrm{P}_\Lambda^{(s)}$).

Local minimality of the first solution in $S_0^1(\Omega)$

To prove the local minimality of $u_\lambda := \arg\min_M I_\lambda$ (wrt to the S_0^1-topology), we suppose there exists $u_j \to u_\lambda$ in $S_0^1(\Omega)$ such that

$$I_\lambda(u_j) < I_\lambda(u_\lambda) \quad \forall j.$$

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$$I_\lambda(u_j) < I_\lambda(u_\lambda) \quad \forall j.$$

We introduce $v_j := \min(\underline{u}, \max(u_j, \overline{u}))$, which is in M for every j, and the two sets $\overline{S}_j := \{u_j > \overline{u}\}$ and $\underline{S}_j := \{u_j < \underline{u}\}$. First we prove

$$\lim_{j\to\infty} |\overline{S}_j| = \lim_{j\to\infty} |\underline{S}_j| = 0,$$

and then we get

$$I_\lambda(u_j) \geq I_\lambda(v_j) \geq I_\lambda(u_\lambda),$$

contradiction.

For the evaluations we followed the approach of [Hai03] in the singular case, and the approach of [Ala99] in the concave-convex case.

Second solution

The approach

We adapt Tarantello's strategy [Tar92]. Recall that the first solution u_λ is a local minimum in $S_0^1(\Omega)$, *i.e.* there exists r_0 such that $I_\lambda(u_\lambda) \leq I_\lambda(u)$ for every $u \in S_0^1(\Omega)$ with $\|u - u_\lambda\|_{S_0^1(\Omega)} < r$.

Then, we restrict to $H_\lambda := \{u \geq u_\lambda \text{ a.e. in } \Omega\} \subset S_0^1(\Omega)$, and we have two cases,

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Then, we restrict to $H_\lambda := \{u \geq u_\lambda \text{ a.e. in } \Omega\} \subset S_0^1(\Omega)$, and we have two cases,

(i) $\forall r \in (0, r_0)$, $\inf_{\|u-u_\lambda\|=r} I_\lambda(u) = I_\lambda(u_\lambda)$,

(ii) $\exists r \in (0, r_0)$ s.t. $\inf_{\|u-u_\lambda\|=r} I_\lambda(u) > I_\lambda(u_\lambda)$.

We treat them, separately, by means of Ekeland's Variational Principle.

First sub-case

Consider a succession $\{u_k\} \subset H_\lambda$ such that $\|u_k - u_\lambda\| = r$ for every k and moreover

$$I_\lambda(u_k) \to I_\lambda(u_\lambda) \text{ for } k \to +\infty.$$

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$$X_\lambda := \{r - \overline{r} \leq \|u - u_\lambda\| \leq r + \overline{r}\} \subset H_\lambda$$

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By applying EVP, for any k there exists $w_k \in X_\lambda$ such that

1. $I_\lambda(w_k) \leq I_\lambda(u_k) \leq I_\lambda(u_\lambda) + \varepsilon_k$
2. $\|w_k - u_k\| \leq \frac{1}{k}$
3. $I_\lambda(w_k) \leq I_\lambda(u) + \frac{\varepsilon_k}{1/k} \|w_k - u\|$ for all $u \in X_\lambda$

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3. $I_\lambda(w_k) \leq I_\lambda(u) + \frac{1}{k} \|w_k - u\|$ for all $u \in X_\lambda$

First sub-case

The sequence w_k is bounded in $S^1_0(\Omega)$ by definition, therefore there exists an element $w_\lambda \in S^1_0(\Omega)$ such that (as $k \to +\infty$, up to sub-sequence)

- $w_k \rightharpoonup w_\lambda$ in $S^1_0(\Omega)$,
- $w_k \to w_\lambda$ strongly in $L^p(\Omega),\ \forall 1 \leq p < 2^*_Q$,
- $w_k \to w_\lambda$ pointwise a.e. in Ω.

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- $w_k \to w_\lambda$ strongly in $L^p(\Omega), \ \forall 1 \leq p < 2_Q^*$,
- $w_k \to w_\lambda$ pointwise a.e. in Ω.

We prove that w_λ is a solution for ($\mathsf{P}_\lambda^{(s)}$) and moreover $\|w_\lambda - u_\lambda\|_{S_0^1(\Omega)} = r$, thus $w_\lambda \not\equiv u_\lambda$, it's another solution.

Second sub-case

We consider the space of continuous curves $C([0,1], H_\lambda)$ endowed with the max-distance

$$d(\eta, \eta') = \max_{t \in [0,1]} \|\eta(t) - \eta'(t)\|_{S^1_0(\Omega)} .$$

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1. $\eta(0) = u_\lambda$
2. $I_\lambda(\eta(1)) < I_\lambda(u_\lambda)$

$\Rightarrow$ Necessarily, $\|\eta(1) - u_\lambda\|_{S_0^1(\Omega)} > r_0$

Second sub-case

We consider the space of continuous curves $C([0,1], H_\lambda)$ endowed with the max-distance

$$d(\eta, \eta') = \max_{t\in[0,1]} \|\eta(t) - \eta'(t)\|_{S_0^1(\Omega)} .$$

In particular we consider the subset $\Gamma_\lambda \subset C([0,1], H_\lambda)$ of curves such that,

1. $\eta(0) = u_\lambda$
2. $I_\lambda(\eta(1)) < I_\lambda(u_\lambda)$

$\Rightarrow$ Necessarily, $\|\eta(1) - u_\lambda\|_{S_0^1(\Omega)} > r_0$

First, we prove that $\Gamma_\lambda \neq \emptyset$, then, by estimating

$$\ell_0 := \inf_{\eta\in\Gamma_\lambda} \max_{t\in[0,1]} I_\lambda(\eta(t)),$$

we get the second solution.

Γ_λ is non-empty

We build a curve in Γ_λ with the help of the "subRiemannian-Aubin-Talenti" functions we introduced before.

Lemma

There exists $\varepsilon_0 > 0$ and $R \geq 1$ such that

$$\begin{aligned} &I_\lambda(u_\lambda + tRU_\varepsilon) < I_\lambda(u_\lambda) && \text{for all } \varepsilon \in (0, \varepsilon_0),\ t \geq 1, \\ &I_\lambda(u_\lambda + tRU_\varepsilon) < I_\lambda(u_\lambda) + \tfrac{1}{Q} S_{\mathbb{G}}^{Q/2} && \text{for all } \varepsilon \in (0, \varepsilon_0),\ t \in [0, 1]. \end{aligned}$$

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- $U_\varepsilon = \varphi T_\varepsilon$
- T_ε attains $S_{\mathbb{G}} = \inf \left(\| |\nabla_{\mathbb{G}} T| \|_2^2 \,/\, \|T\|_{2^*_Q}^2 \right)$
- in particular, $\| |\nabla_{\mathbb{G}} T_\varepsilon| \|_2^2 = \|T_\varepsilon\|_{2^*_Q}^{2^*_Q} = S_{\mathbb{G}}^{Q/2}$

Γ_λ is non-empty

By what we know on U_ε, we build the estimate

$$I_\lambda(u_\lambda+tRU_\varepsilon) < I_\lambda(u_\lambda)+\frac{Bt^2R^2}{2}-\frac{At^{2^*_Q}R^{2^*_Q}}{2^*_Q}-(tR)^{2^*_Q-1}K\,\varepsilon^{\frac{Q-2}{2}}+o\left(\varepsilon^{\frac{Q-2}{2}}\right),$$

where $A = \|T_\varepsilon\|_{2^*_Q}$, $B = \||\nabla_{\mathbb{G}} T_\varepsilon|\|_2^2 = S_{\mathbb{G}}^{Q/2}$ and $K > 0$.

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By what we know on U_ε, we build the estimate

$$I_\lambda(u_\lambda + tRU_\varepsilon) < I_\lambda(u_\lambda) + \frac{Bt^2R^2}{2} - \frac{At^{2^*_Q}R^{2^*_Q}}{2^*_Q} - (tR)^{2^*_Q-1} K\,\varepsilon^{\frac{Q-2}{2}} + o\left(\varepsilon^{\frac{Q-2}{2}}\right),$$

where $A = \|T_\varepsilon\|_{2^*_Q}$, $B = \||\nabla_{\mathbb{G}} T_\varepsilon|\|_2^2 = S_{\mathbb{G}}^{Q/2}$ and $K > 0$.

This allows to conclude because we prove that the left side is always bounded by

$$I_\lambda(u_\lambda) + \left(\frac{1}{2} - \frac{1}{2^*_Q}\right) \frac{B^{\frac{2^*_Q}{2^*_Q-2}}}{A^{\frac{2}{2^*_Q-2}}} - O(\varepsilon^{\frac{Q-2}{2}}) < I_\lambda(u_\lambda) + \frac{1}{Q} S_{\mathbb{G}}^{Q/2},$$

for ε sufficiently small.

The estimate

The key estimate of I_λ develops as follow,

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla_G u|^2 - \frac{1}{2_Q^*}\int_\Omega u^{2_Q^*} - \frac{\lambda}{1+s}\int_\Omega u^{1+s}$$

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$$\begin{aligned}
I_\lambda(u_\lambda + tRU_\varepsilon) = & \frac{1}{2}\int |\nabla_{\mathbb{G}} u_\lambda|^2 - \frac{1}{2^*_Q}\int u_\lambda^{2^*_Q} - \frac{\lambda}{1+s}\int u_\lambda^{1+s} \\
& + tR\left(\int \langle \nabla_{\mathbb{G}} u_\lambda, \nabla_{\mathbb{G}} U_\varepsilon\rangle_{\mathfrak{g}_1} - \int u_\lambda^{2^*_Q-1}U_\varepsilon - \lambda\int u_\lambda^s U_\varepsilon\right) \\
& - \frac{1}{2^*_Q}\left(\int (u_\lambda + tRU_\varepsilon)^{2^*_Q} - \int u_\lambda^{2^*_Q}\right) + tR\int u_\lambda^{2^*_Q-1}U_\varepsilon \\
& + \frac{t^2R^2}{2}\int |\nabla_{\mathbb{G}} U_\varepsilon|^2 \\
& - \frac{\lambda}{1+s}\left(\int (u_\lambda + tRU_\varepsilon)^{1+s} - \int u_\lambda^{1+s}\right) + \lambda\, tR\int u_\lambda^s U_\varepsilon
\end{aligned}$$

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& + tR \Big(\boxed{\text{This is } 0} \int u_\lambda^s U_\varepsilon \Big) \\
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$$+\, tR\Big(\quad \cdots \quad \int u_\lambda^s U_\varepsilon \Big)$$

This is 0

$$- \frac{1}{2^*_Q}\left(\int (u_\lambda + tRU_\varepsilon)^{2^*_Q} - \int u_\lambda^{2^*_Q}\right) + tR\int u_\lambda^{2^*_Q - 1} U_\varepsilon$$

$$+\, \frac{t^2R^2}{2} \quad \cdots$$

This is $B + o\left(\varepsilon^{Q-2}\right)$

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$$+ tR \Big(\quad \text{This is } 0 \quad u_\lambda^s U_\varepsilon \Big)$$

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$$+ \frac{t^2 R^2}{2} \quad \text{This is } B + o\left(\varepsilon^{Q-2}\right)$$

$$- \frac{\lambda}{1+} \quad R\int u_\lambda^s U_\varepsilon$$

This is $o\left(\varepsilon^{\frac{Q-2}{2}}\right)$, estimated by Taylor expansion and considerations on u

The key estimate of I_λ develops as follow,

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$$+ tR \Big(\quad \int u_\lambda^s U_\varepsilon \Big)$$

This is 0

$$\boxed{- \frac{1}{2^*_Q}\left(\int (u_\lambda + tRU_\varepsilon)^{2^*_Q} - \int u_\lambda^{2^*_Q}\right) + tR\int u_\lambda^{2^*_Q - 1} U_\varepsilon}$$

This is

$$-\frac{A(tR)^{2^*_Q}}{2^*_Q} + O\left(\varepsilon^{\frac{Q-2}{2}}\right),$$

the key role in this estimate is played by the convolution defined on $\mathbb{G}$ via its homogeneous Lie group structure.

$$\int u_\lambda^s U_\varepsilon$$

Convergence to the minimax level

Now that we know $\Gamma_\lambda \neq \emptyset$, we can prove that there exist sequences $\{t_k\} \subset [0,1]$ and $\{\eta_k\} \subset \Gamma_\lambda$ such that

1. $I_\lambda(\eta_k(t_k)) = \max_{s\in[0,1]} I_\lambda(\eta_k(s))$

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2. $I_\lambda(v_k := \eta_k(t_k)) \to \ell_0$ for $k \to +\infty$
3. there exists $C > 0$ such that for all $w \in H_\lambda$,

$$\int_\Omega \langle \nabla_{\mathbb{G}} v_k, \nabla_{\mathbb{G}}(w - v_k)\rangle - \lambda \int_\Omega v_k^s (w - v_k) - \int_\Omega v_k^{2_Q^*-1}(w - v_k) \geq -\frac{C}{k}(1 + \|w\|_{S_0^1(\Omega)})$$

Existence of the second solution

Once again we obtain that v_k is bounded, therefore $v_k \rightharpoonup v_\lambda$ a solution of $(\mathsf{P}_\lambda^{(s)})$.

Moreover, $I_\lambda(u_\lambda) < I_\lambda(v_\lambda) < I_\lambda(u_\lambda) + S_{\mathbb{G}}^{Q/2}$, therefore $v_\lambda \not\equiv u_\lambda$.

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Theorem (Biagi, G., Vecchi, 2026)

If Ω is a bounded and connected open set of a Carnot group $\mathbb{G}$, then there exists $\Lambda > 0$ such that $(\mathsf{P}_\lambda^{(s)})$ has at least one solution u_λ for every $\lambda \in (0, \Lambda]$ and no solution for $\lambda > \Lambda$. Moreover, $(\mathsf{P}_\lambda^{(s)})$ has a second solution v_λ for every $\lambda \in (0, \Lambda)$.

The end

Thank you for your attention!

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