Weak Harnack and rigidity results for the anisotropic Trudinger's equation

joint work with S. Ciani and I.I. Skrypnik

Mattia Galeotti (UniBO)
Università di Bologna
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Quasilinear equations of p-Laplacian type

Let $E \subset \mathbb{R}^N$ open and $E_T := E \times (0, T]$.

We start by considering the parabolic equation with respect to the p-Laplacian

$$u_t - \operatorname{div}\left(|Du|^{p-2} Du\right) = 0, \quad p > 1, \quad \text{weakly in } E_T,$$

Quasilinear equations of p-Laplacian type

Let $E \subset \mathbb{R}^N$ open and $E_T := E \times (0, T]$.

We work with a class of homogeneous parabolic equations, given a vector field $A\colon E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$,

$$u_t - \mathrm{div} A(x, t, u, Du) = 0, \quad \text{weakly in } E_T.$$

Here A must respect the following a.e. in E_T,

$$\begin{cases} A(x, t, u, Du) \cdot Du \geq C_0 |Du|^p \\ |A(x, t, u, Du)| \leq C_1 |Du|^{p-1} \end{cases}$$

with $p > 1$ and $C_0, C_1 > 0$.

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with $p > 1$ and $C_0, C_1 > 0$.

A function $u \in C_{\mathrm{loc}}(0, T; L^2_{\mathrm{loc}}(E)) \cap L^p_{\mathrm{loc}}(0, T; W^{1,p}_{\mathrm{loc}}(E))$ is a local, weak sub(super)-solution, if $\forall\, K \subset E$ compact, and $\forall\, [t_1, t_2] \subset (0, T]$,

$$\int_K u\varphi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K (-u\varphi_t + A \cdot D\varphi) \, dxdt \leq (\geq)\, 0$$

$\forall$ testing functions $\varphi \in W^{1,2}_{\mathrm{loc}}(0, T; L^2(K)) \cap L^p_{\mathrm{loc}}(0, T; W^{1,p}_0(K))$.

The Harnack inequality for non-degenerate homogeneous equations ($p = 2$)

Consider the case $p = 2$, this is the first generalization of the heat equation.

We introduce

$$\begin{array}{lll} \text{Cubes} & K_r(\overline{x}) & = \{x \in \mathbb{R}^N : \ |x_i - \overline{x}_i| < r, \ \forall i\} \\ \text{Cylinders} & Q^+_{r,z}(\overline{x}, \bar{t}) & = K_r(\overline{x}) \times \left(\bar{t} + (0, z]\right). \end{array}$$

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Theorem (See [Mos64])

Let $u\geq 0$ be a continuous, local, weak solution to the non-degenerate equation in E_T. Then, $\exists\,\gamma>0$ depending only on the data (N, C_0, C_1) such that

$$\gamma^{-1}\sup_{K_\rho(x_0)} u(\cdot,\, t_0-\rho^2)\leq u(x_0,t_0)\leq \gamma \inf_{K_\rho(x_0)} u(\cdot, t_0+\rho^2),$$

with the condition that $Q^{\pm}_{4\rho,(4\rho)^2}\subset E_T$.

Sketch of the proof

We will use the notation

$$P(K;\, B) = \frac{|K \cap B|}{|K|}$$

$$P_t(K;\, u \geq k) = P(K;\, \{x :\ u(x,t) \geq k\}).$$

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The key ingredient is the expansion of positivity.

Lemma

If for some $(x_0, t_0) \in E_T$*, and* $\rho > 0$ *we have*

$$P_{t_0}(K_\rho(x_0);\, u \geq k) \geq \alpha$$

for some $k > 0$ *and* $\alpha \in (0,1)$*, then there exists* $\eta, \delta \in (0,1)$ *depending only on the data and* α *such that*

$$u \geq \eta k \quad \text{in } K_{2\rho}(x_0) \times \left(t_0 + \left(\frac{\delta}{2}\rho^2, \delta\rho^2\right]\right).$$

Sketch of the proof

Observe that by the expansion of positivity

$$P_{t_0}(K_\rho(x_0);\, u \geq k) \geq \alpha \;\Rightarrow\; P_{t_0+\theta\rho^2}(K_{2\rho}(x_0);\, u \geq \eta k) = 1,$$

we get an iteration.

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Therefore, from $P_{t_0}(K_r(x_0);\, u \geq k) \geq \tilde{\alpha}$ we get

$$\inf_{K_\rho(x_0)} u(\cdot, t_0 + \lambda\rho^2) \geq \tilde{\eta} k \left(\frac{r}{\rho}\right)^\beta,$$

thus proving that the left hand side is bounded from below by $\overline{m} > 0$.

We treat the degenerate case, where $p > 2$.

Theorem

Let $u \geq 0$ *a continuous, local, weak solution to the degenerate equation in* E_T*. Then,* $\exists\, c, \gamma > 0$ *depending only on the data and such that*

$$\gamma^{-1} \sup_{K_\rho(x_0)} u(\cdot,\, t_0 - \theta\rho^p) \leq u(x_0, t_0) \leq \gamma \inf_{K_\rho(x_0)} u(\cdot, t_0 + \theta\rho^p),$$

with the condition that $Q^{\pm}_{4\rho,\theta(4\rho)^p}(x_0, t_0) \subset E_T$.

In this case the inequality is intrinsic to the solution, in particular

$$\theta = \left(\frac{c}{u(x_0, t_0)}\right)^{p-2}$$

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The "intrinsic" cylinders are natural because they correspond to the re-scalings of the solution that are compatible with the equation. For a wider analysis see [DiB93, DGV12]

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$$P_{t_0}(K_\rho(x_0);\, u \geq k) \geq \alpha$$

for some $k > 0$ *and* $\alpha \in (0,1)$*, then there exists* $\eta, \delta \in (0,1)$ *and* $b > 1$*, depending only on the data and* α *such that*

$$u \geq \eta k \quad \text{in } K_{2\rho}(x_0)$$

for all times

$$t \in t_0 + \left[\left(\frac{b}{\eta k}\right)^{p-2} \frac{\delta}{2}\rho^p, \left(\frac{b}{\eta k}\right)^{p-2} \delta\rho^p \right].$$

Barenblatt profile

The obstruction to a non-intrinsic Harnack is explicit. For $u_t = \mathrm{div}(|Du|^{p-2}Du)$, $p > 2$, the fundamental solution is

$$\mathcal{B}_p(x,t) = \frac{1}{t^{N/\lambda}}\left[\left(1 - b\left(\frac{|x|}{t^{1/\lambda}}\right)^{\frac{p}{p-1}}\right)_+\right]^{\frac{p-1}{p-2}}, \quad \lambda = N(p-2)+p.$$

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- the compact support expands like $t^{1/\lambda}$, *i.e.* the natural cylinders depend on the size of u;
- hence no Harnack in $K_\rho \times (t_0, t_0 + \rho^p)$.
  The waiting time $\theta\rho^p$ is unavoidable.

The anisotropic Trudinger's equation

The equation

Leibenson (1930), turbulent filtration of a gas in a porous medium:

$$\partial_t u^m = c\,\Delta_p u, \qquad \left(m+1 = p = \tfrac{3}{2}\right).$$

For $m = p - 1$ the equation is *homogeneous and doubly nonlinear*, and Moser's method yields a Harnack estimate [Tru68].

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$$\partial_t v = \overline{c}\,\mathrm{div}\left(v^q\,|\nabla v|^{p-2}\nabla v\right) \qquad \text{with } \overline{c} = \tfrac{c}{m^{p-1}},\ q = \tfrac{(p-1)(1-m)}{m}.$$

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We are interested in the case $m = p-1$,

$$\partial_t v = \mathrm{div}\left(v^{2-p}\,|\nabla v|^{p-2}\nabla v\right).$$

The equation

Different materials diffuse with different power laws along different directions. Our equation is

$$\partial_t u - \sum_{i=1}^{N} \partial_i \left(u^{2-p_i} |\partial_i u|^{p_i - 2} \partial_i u \right) = 0, \qquad u \geq 0,$$

where $1 < p_1 \leq \cdots \leq p_N$ and we denote by p the harmonic mean,

$$\frac{1}{p} = \frac{1}{N} \sum_i \frac{1}{p_i}, \quad \text{with } p < N.$$

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The equation is homogeneous in u, therefore it has no intrinsic geometry.

Weak Harnack inequality

Let's introduce

Cubes	$K_r(\overline{x})$	$= \{x \in \mathbb{R}^N : \ |x_i - \overline{x}_i| < r, \ \forall i\}$
Anisotropic cubes	$\mathcal{K}_r(\overline{x})$	$= \left\{x \in \mathbb{R}^N : \ |x_i - \overline{x}_i| < r^{\frac{p}{p_i}}, \ \forall i\right\}$
Cylinders	$Q^+_{r,z}(\overline{x}, \bar{t})$	$= \mathcal{K}_r(\overline{x}) \times \left(\bar{t} + (0, z]\right)$

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Theorem (Ciani, G., Skrypnik, 2026)

If for some $\rho > 0$, $u \geq 0$ *is a local weak supersolution of the equation in* $Q^+_{4\rho,(4\rho)^p}(x_0, t_0)$

then $\exists\, \delta_0, q \in (0,1), \gamma > 1$ *depending only on the data* (N, p_i)*, such that*

$$\left(\fint_{\mathcal{K}_\rho(x_0)} u^q(x,s)\, dx \right)^{\frac{1}{q}} \leq \gamma \inf_{\mathcal{K}_{2\rho}(x_0)} u(\cdot, t),$$

for all $t \in t_0 + \left(\frac{\delta_0}{2}\rho^p, \delta_0 \rho^p\right)$.

Energy estimates

Let $u \geq 0$ be a local weak sub(super)-solution, ζ a cutoff vanishing on $\partial\mathcal{K}_r(x_0)$, $k > 0$, $m = \frac{1}{p_N - 1}$. Then

$$\sup_{t \in t_0 + (-z,0)} \int_{\mathcal{K}_r(x_0)} g_\pm(u^m, k^m)\zeta^{p_N} + \frac{1}{\gamma}\sum_i \iint_{Q^-_{r,z}(x_0,t_0)} u^{\frac{p_N - p_i}{p_N - 1}} \left|D_i(u^m - k^m)_\pm\right|^{p_i} \zeta^{p_N}$$
$$\leq \int_{\mathcal{K}_r(x_0)\times\{t_0 - z\}} g_\pm(u^m, k^m)\zeta^{p_N} + \gamma \iint_{Q^-_{r,z}(x_0,t_0)} g_\pm(u^m, k^m)|\zeta_t|$$
$$+ \gamma \sum_i \iint_{Q^-_{r,z}(x_0,t_0)} u^{\frac{p_N - p_i}{p_N - 1}} (u^m - k^m)_\pm^{p_i} |D_i\zeta|^{p_i} .$$

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The doubly nonlinear nature sits in the terms

$$g_\pm(u^m, k^m) := \pm \frac{1}{m} \int_{k^m}^{u^m} s^{\frac{1}{m} - 1}(s - k^m)_\pm \, ds$$

They respect the following estimate for some $\gamma > 0$

$$\gamma^{-1}(k^m + u^m)^{\frac{1}{m} - 1}(u^m - k^m)_\pm^2 \leq g_\pm \leq \gamma (k^m + u^m)^{\frac{1}{m} - 1}(u^m - k^m)_\pm^2$$

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Everything below follows from these estimates alone

Lemma (measure information travels in time)

Let $u \geq 0$ *local weak super-solution to the equation in* $Q = \mathcal{K}_8(x_0) \times (t_0 + (0,1))$, *and* $\exists\, \alpha \in (0,1), k > 0$ *such that*

$$P_{t_0}(\mathcal{K}_1(x_0);\, u \geq k) \geq \alpha.$$

Then, $\exists\, \eta_1, \delta_1 \in (0,1)$ *depending only on the data and* α, *such that*

$$P_t(\mathcal{K}_1(x_0);\, u \geq \eta_1 k) \geq \frac{\alpha}{4} \quad \forall t \in t_0 + (0, \delta_1).$$

Here the dependence is explicit

$$\delta_1 \approx \frac{\alpha^{p_N+1}(1-\alpha)}{\gamma} \quad \text{and} \quad \eta_1 \approx \frac{\alpha}{\gamma}$$

for some $\gamma > 0$ depending only on the data.

Refined expansion of positivity

Lemma (Expansion of positivity, De Giorgi-type, see [CHSS26])

Let $u \geq 0$ local weak super-solution, and $\exists\, r, k > 0$, $\alpha \in (0,1)$ with

$$P_{t_0}(\mathcal{K}_r(x_0);\, u \geq k) \geq \alpha.$$

Then $\exists\, \eta, \delta \in (0,1)$ depending only on the data and α, such that

$$u(\cdot, t) \geq \eta k \quad \text{in } \mathcal{K}_{2r}(x_0), \qquad \forall t \in t_0 + \left(\tfrac{\delta}{2} r^p, \delta r^p\right).$$

The non-explicit dependence $\eta = \eta(\alpha)$ makes it hard to iterate.

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Then $\exists\, \eta, \delta \in (0,1)$ *and* $d > 1$ *depending only on the data ~~and~~* $\cancel{\alpha}$, *such that*

$$u(\cdot, t) \geq \eta \alpha^d k \quad \text{in } \mathcal{K}_{2r}(x_0), \qquad \forall t \in t_0 + \left(\tfrac{\delta}{2} r^p, \delta r^p\right).$$

The dependence on α is now a power with exponent

$$d = 1 + \left(2 + \left(\frac{p_N + 1}{p_1}\right) \log_2(\eta^{-1})\right)$$

This will make the layer-cake integral converge in a moment.

Sketch of proof for the weak Harnack inequality

We set $(x_0, t_0) = (0, 0)$ and

$$I := \inf_{\mathcal{K}_{2\rho} \times \left(\frac{\delta_0}{2}\rho^p, \delta_0 \rho^p\right)} u.$$

and we use the layer-cake formula

$$\int_{\mathcal{K}_\rho} u^q(x,0)\,dx \le q \int_I^\infty \left|\{u(\cdot,0) > k\} \cap \mathcal{K}_\rho\right| k^{q-1}\,dk + I^q |\mathcal{K}_\rho|.$$

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We apply the refined expansion with $\alpha(k) := P_0(\mathcal{K}_\rho; u > k)$

We get $\alpha(k)^d \le \frac{I}{\eta k}$, therefore if we choose $q < \frac{1}{d}$ then the integral converges and gives

$$\int_{\mathcal{K}_\rho} u^q(\cdot, 0) \le \gamma I^q |\mathcal{K}_\rho|.$$

$\square$

Refined L^∞ bound

In order to prove a (non-weak) Harnack inequality, we refine the L^∞ bound for a subsolution.

Lemma

If $(x_0, t_0) \in E_T$ and $\theta, \rho > 0$ such that $Q^-_{4\rho,\theta(4\rho)^p}(x_0, t_0) \subset E_T$.

If $u \geq 0$ is a subsolution, then

$$\sup_{Q^-_{\rho,\theta\rho^p}(x_0,t_0)} u \leq \gamma\, \theta^{\frac{N(1-p_N)}{p\, p_N}} \left(\sup_{t \in t_0 + (-\theta\rho^p, 0)} \fint_{\mathcal{K}_\rho(x_0)} u^q(x,t)\, dx \right)^{\frac{1}{q}}$$

Through a De Giorgi-type lemma, we can prove the following result of semicontinuity

Lemma

Every $u \geq 0$ local weak subsolution has an upper-semicontinuous representative u^*.

The semicontinuous regularisation is defined as

$$u^*(x,t) := \operatorname{ess} \limsup_{(y,s)\to(x,t)} u(y,s) = \lim_{\rho\to 0^+} \operatorname*{ess\,sup}_{Q_\rho(x,t)^\pm} u$$

Weak Harnack implies Harnack

Theorem (Ciani, G., Skrypnik, 2026)

Suppose $(x_0, t_0) \in E_T$ *and* $\theta, \rho > 0$ *such that* $Q^{\pm}_{4\rho, \frac{\delta_0}{4}(4\rho)^p} \subset E_T$, *then*

$$\sup_{Q_S} u \leq \gamma \inf_{Q_I} u$$

with

$$Q_S = Q^{-}_{\rho, \frac{\delta_0}{4}\rho^p}(x_0, t_0), \quad Q_I = \mathcal{K}_{\rho}(x_0) \times \left(t_0 + \left(\frac{\delta_0}{2}\rho^p, \frac{3\delta_0}{4}\rho^p\right)\right)$$

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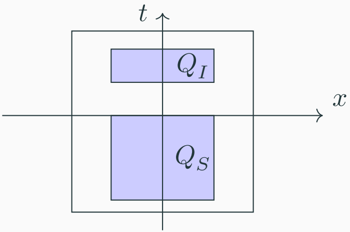

Sketch of the proof

By the refined L^∞ bound,

$$\gamma^{-1}\sup_{Q_S} u \le \left(\sup_{t\in t_0+(-\theta\rho^p,0)} \fint_{\mathcal{K}_\rho(x_0)} u^q(x,t)\,dx \right)^{\frac{1}{q}}.$$

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We consider the upper-semicontinuous representative of u, then $t \mapsto \|u(\cdot,t)\|_{q,\mathcal{K}_\rho}$ attains its $\sup$ at some $\tau \in [t_0 - \theta\rho^p, t_0]$.

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By the refined L^∞ bound,

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We consider the upper-semicontinuous representative of u, then $t \mapsto \|u(\cdot,t)\|_{q,\mathcal{K}_\rho}$ attains its sup at some $\tau \in [t_0 - \theta\rho^p, t_0]$.

We apply the weak Harnack to the right hand side to get

$$\sup_{Q_S} u \leq \gamma \inf_{Q_I} u.$$

with $Q_I = \mathcal{K}_\rho(x_0) \times \left(\tau + \left(\frac{\delta_0}{2}\rho^p, \delta_0\rho^p\right)\right)$,

Sketch of the proof

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with $Q_I = \mathcal{K}_\rho(x_0) \times \left(\tau + \left(\frac{\delta_0}{2}\rho^p, \delta_0\rho^p\right)\right)$, up to re-scaling the time interval, we can drop the dependency on τ. $\square$

Sub-potential lower bounds and time-decay

Theorem (Ciani, G., Skrypnik, 2026)

Let $u > 0$ be a local weak solution in $\mathbb{R}^{N+1}$. Then $\exists\, \gamma > 0$ depending only on the data such that, for $t > s$,

$$u(x,t) \geq \gamma^{-1} u(y,s)\, \mathcal{B}(x,t,y,s),$$

where

$$\mathcal{B} := \exp\left(-\sum_i \left(\frac{|x_i - y_i|^{p_i}}{t-s} \right)^{\frac{1}{p_i - 1}} \right).$$

In a bounded E_T the same chain gives a decay in time,

$$u(x,t) \geq \gamma^{-1} \left(\frac{t}{s} \right)^{A} u(y,s)\, \mathcal{B}(x,t,y,s).$$

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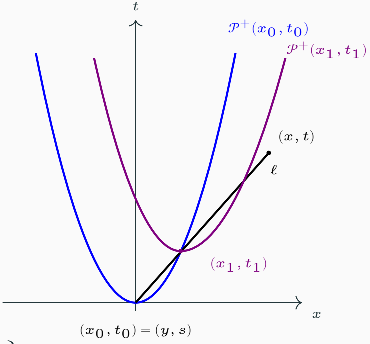
This result applies to the entire class of functions respecting the energy estimates.

Sketch of proof

Harnack chain. From the Harnack inequality, if we set $\theta = \frac{3\delta_0}{4}$, we get

$$u(x_0, t_0) \leq C \inf_{\mathcal{K}_\rho(x_0)} u(\cdot, t_0 + \theta\rho^p).$$

The point (x, t) is reached along a segment through n points (x_j, t_j), each inside the forward paraboloid of the previous one,



$$\mathcal{P}^+(\overline{x}, \overline{t}) = \left\{ (x, t) : \theta |\overline{x}_i - x_i|^{p_i} \leq t - \overline{t} \quad \forall i \right\},$$

so that $u(x_j, t_j) \leq C u(x_{j+1}, t_{j+1})$ and $\dfrac{u(x,t)}{u(y,s)} \leq C^n$, with

$$n \approx 1 + \frac{s}{\theta\rho^p} + \sum_i \frac{|y_i|}{(8\rho)^{1/p_i}} + \left(\frac{\theta |y_i|^{p_i}}{s} \right)^{\frac{1}{p_i - 1}}$$

A Barenblatt-type profile

The function $\mathcal{B}$ is a natural candidate as a solution for the prototype equation $u_t = \sum_i \partial_i \left(u^{2-p_i}|\partial_i u|^{p_i-2}\partial_i u\right)$.

Indeed,

$$\mathcal{B}(x,t) = D\,t^{\alpha} \exp\left(-\sum_i c_i \left(\frac{|x_i|^{p_i}}{t}\right)^{\frac{1}{p_i-1}}\right)$$

is a solution of the equation, for

$$c_i = \left((p_i-1)^{p_i-1} p_i^{-p_i}\right)^{\frac{1}{p_i-1}}, \qquad \alpha = -\sum_i \left(\frac{c_i p_i}{p_i-1}\right)^{p_i-1}$$

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Recall that for the isotropic Trudinger's equation, the Barenblatt solution is

$$\mathcal{B}_p(x,t) = \frac{1}{t^{N/p}} \exp\left(-\xi\left(\frac{|x|}{t^{1/p}}\right)^{\frac{p}{p-1}}\right), \quad \xi = (p-1)p^{-\frac{p}{p-1}}$$

More general structures

Weak Harnack, Harnack and the sub-potential bounds only used the energy estimates. Hence they hold for

$$u_t - \sum_i D_i A_i(x,t,u,Du) = 0 \quad \text{weakly in } E_T,$$

with A_i Carathéodory functions satisfying

$$\begin{cases} A_i(x,t,s,\xi)\,\xi_i \geq K_1\, s^{2-p_i}|\xi_i|^{p_i} \\ |A_i(x,t,s,\xi)| \leq K_2\, s^{2-p_i}|\xi_i|^{p_i-1} \end{cases} \qquad \forall (s,\xi) \in \mathbb{R}_+ \times \mathbb{R}^N.$$

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Such operators need not satisfy a comparison principle.

Open. Continuity of bounded solutions to the anisotropic p-Laplacian $u_t - \sum_i \partial_i(|\partial_i u|^{p_i-2}\partial_i u) = 0$.

The end

Thank you for your attention!

S. Ciani, E. Henriques, M. O. Savchenko, and I. I. Skrypnik.
Qualitative properties of solutions to parabolic anisotropic equations, part II — The anisotropic Trudinger's equation.
Journal of the London Mathematical Society, 114(1):e70643, 2026.

Ennio De Giorgi.
Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari.
Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat., Vol. 3:25–43, 1957.

Emmanuele DiBenedetto, Ugo Gianazza, and Vincenzo Vespri.
Harnack's Inequality for Degenerate and Singular Parabolic Equations.
Springer Monographs in Mathematics. Springer, New York, NY, 2012.

Emmanuele DiBenedetto.
Degenerate Parabolic Equations.
Universitext. Springer, New York, NY, 1993.

Jürgen Moser.
A harnack inequality for parabolic differential equations.
Communications on Pure and Applied Mathematics, 17(1):101–134, 1964.

Neil S. Trudinger.
Pointwise estimates and quasilinear parabolic equations.
Communications on Pure and Applied Mathematics, 21(3):205–226, 1968.