SR geodesics

1

12/1/26

* First idea: controllability and (linearized) control systems

* Differential Geometry preliminaries

Controllability

Consider an ODE $\dot{q} = f(q, u)$

$q : \mathbb{R} \to \mathbb{R}^n$

$u : \mathbb{R} \to \mathbb{R}^m$

$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

q <u>state</u>

u <u>control</u>

<u>Definition:</u> the system is **controllable** if $\forall q_0, q_1 \in \mathbb{R}^n$ $\exists u(t)$ st the solution of

$$\begin{cases} \dot{q}(t) = f(q(t), u(t)) \\ q(0) = q_0 \end{cases}$$

passes through q_1, $\exists T$ st $q(T) = q_1$

How do we know if a system is controllable?

First idea: look at the linearization

$$\dot{b} = \frac{\partial f}{\partial x}(\bar{q},\bar{u})\cdot b + \frac{\partial f}{\partial u}(\bar{q},\bar{u})\cdot v$$

$b = \delta q$ lin. of q

$v = \delta u$ lin. of u

GENERAL CONTROLLABILITY $\Longrightarrow$ CONTROLLABILITY OF LINEARIZED SYSTEM

$\not\Longleftarrow$

NO

Example: $\mathbb{R}^3$

$$\begin{cases} \dot{x} = u_1 \\ \dot{y} = u_2 \\ \dot{z} = x u_2 \end{cases} \qquad q = (x, y, z)$$

equivalently $\dot{q} = u_1(t) X_1 + u_2(t) X_2$ where $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ 1 \\ x(t) \end{pmatrix}$

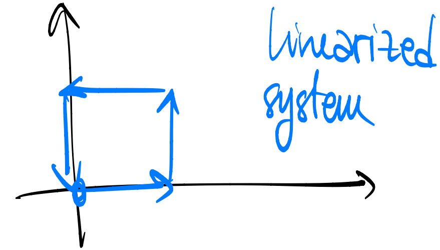
The linearized system in $(0,0)$ is non-controllable,

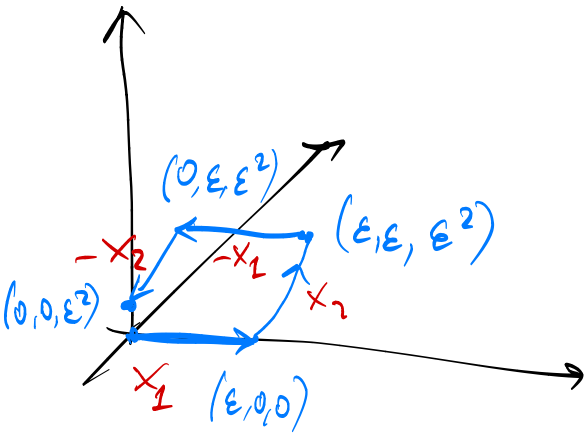
because $\dot{q} \in \langle X_1, X_2 \rangle$ which has dimension $= 2 < 3$

$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

we are going to show that we can reach points on $\{(0,0,z)\}$

$$\left(u_1(t), u_2(t)\right) = \begin{cases} (1,0) & \text{when } t \in [0,\varepsilon) \\ (0,1) & \text{when } t \in [\varepsilon, 2\varepsilon) \\ (-1,0) & \text{when } t \in [2\varepsilon, 3\varepsilon) \\ (0,-1) & \text{when } t \in [3\varepsilon, 4\varepsilon) \end{cases}$$





We have controllability even if the lin. system is non-controllable

time $= 4\varepsilon$

distance $= \varepsilon^2$

T travel time

if we go from $(0,0,0)$ to (x,y,z)

following trajectories that respect the control system

$$\Rightarrow \quad x = \int_0^T u_1(t)\,dt \le T$$

$$y = \int_0^T u_2(t)\,dt \le T$$

$$z = \int_0^T u_2(t)\cdot x(t)\,dt \le T^2$$

$$\frac{1}{3}\left(|x| + |y| + |z|^{1/2}\right) \le T$$

Observation: I can go from 0 to $(x,0,0)$ in time $|x|$

from $(x,0,0)$ to (x,y,xy) in time $|y|$

from (x,y,xy) to (x,y,z) in time $4\sqrt{|z-xy|}$

$$T \leq |x| + |y| + 4\sqrt{|z-xy|} \leq |x| + |y| + 4|z|^{1/2} + 2|x| + 2|y|$$

$$\leq 4\left(|x| + |y| + |z|^{1/2}\right)$$

we just showed $\quad T \sim |x| + |y| + |z|^{1/2}$

Differential Geometry preliminaries

M smooth manifold

if $q \in M$, $C^\infty(M) = \{ f : M \to \mathbb{R} \text{ smooth} \}$, $C^\infty(q) = \{ f \in C^\infty(U)$ st $U \ni q$ neighborhood $\}$

(Equivalent) definitions of tangent space $T_q M$

(1) $T_q M = \{ V : C^\infty(q) \xrightarrow[\text{linear}]{} \mathbb{R}$ st $V(fg) = f(q) \cdot V(g) + g(q) \cdot V(f) \}$

(2) $T_q M = \{ \gamma : (-\varepsilon, \varepsilon) \to M \mid \gamma(0) = q \} \Big/ \Big[\gamma \sim \bar{\gamma}$ if $(f \circ \gamma)'(q) = (f \circ \bar{\gamma})'(q)$ $\forall f \in C^\infty(M) \Big]$

(3) $T_q M = \left\langle \frac{\partial}{\partial x_i} \;\middle|\; i = 1, \dots, n \right\rangle$ $\{x_1, \dots, x_n\}$ loc. chart of M

Sketch of $(1) \equiv (3)$

if V is a derivation we can prove that $V(\text{constant}) = 0$

and we can show that $\frac{\partial}{\partial x_i}$ are derivations for $i = 1, \dots, n$

Now we want to show that

every derivation V can be written as

$$V = \sum_{i=1}^{n} V(x_i) \cdot \frac{\partial}{\partial x_i}$$ (Exercise)

Vector fields

we defined a derivation of $C^\infty(q)$, in general a
derivation is an operator $C^\infty(M) \to C^\infty(M)$
such that $V(fg) = V(f) \cdot g + V(g) \cdot f$

$$\mathrm{Vec}(M) = \{ \text{Derivations } C^\infty(M) \to C^\infty(M) \}$$

This is equivalent to the notion of
tangent bundle $TM \to M$ st T_qM is defined as before

If $X \in \mathrm{Vec}(M)$ then the $\begin{cases} \dot{q}(t) = X(q(t)) \\ q(0) = q_0 \end{cases}$ $q_0 \in M$

has a solution at least locally

$\exists \varepsilon > 0$, $\gamma : (-\varepsilon, \varepsilon) \to M$ st $\gamma(t, q)$ is defined in some neighborhood $(-\varepsilon, \varepsilon) \times U$ where $U \subset M$ is a neighborhood of q_0

$\boxed{\text{The vector field is \underline{complete} when the solution } \gamma(t,q) \text{ exists } \forall t \in \mathbb{R}\ \forall q \in M}$

Flow map

In the case of a complete vector field we can consider the map

$$P_t^X : M \longrightarrow M$$
$$q_0 \longmapsto q(t)$$

where $q(\cdot)$ is the solution of the associated ODE

P_t^X, $P_{s,t}^X$ is smooth in (t,q)

and P_t^X is a 1-parameter subgroup of $\mathrm{Diff}(M)$

$$\begin{cases} P_0 = \mathrm{id} \\ P_t \circ P_s = P_s \circ P_t = P_{t+s} \\ P_t^{-1} = P_{-t} \quad \text{(implied by above)} \end{cases}$$

Sometimes for the flow map we use the notation

$$P_t^X = \exp(tX) = e^{tX}$$

<u>Exercise</u>: $a \in C^\infty(M)$, $a_t : a(e^{tX})$ $\quad a_t : \mathbb{R} \times M \longrightarrow \mathbb{R}$

(i.) $\dot{a}_t = \frac{d}{dt} a(e^{tX}) = Xa$

(ii.) $a_t(q) = a(q) + t \cdot Xa(q) + \dots$ the usual Taylor series

— // —

Non-autonomous vector fields

A na-vf is a family $\{X_t \in \mathrm{Vec}(M)\}_{t\in\mathbb{R}}$

such that

(1) $t \longmapsto X_t(q)$ is measurable $\forall q \in M$

(2) $q \longmapsto X_t(q)$ is smooth $\forall t \in \mathbb{R}$

$\equiv \forall a \in C^\infty(M)$

$(t,q) \mapsto X_t a|_q$

is measurable wrt t
is smooth wrt q

(3) for every system of coordinates on $\Omega \subset M$ and every $K \subset \Omega$ compact and $I \subset \mathbb{R}$ compact interval $\exists\ c(t), k(t) \in L^\infty(I)$ st

$\| X_t(q) \| \leq c(t)$; $\| X_t(q) - X_t(q') \| \leq k(t) \| q - q' \|$ $\quad \forall t \in I$, $\forall q, q' \in K$

Example $X_t(q) = \sum u_i(t)\, X_i(q)$ where $u_i(t)$ control functions and $X_i \in Vec(M)$

the conditions above mean that $u_i(t)$ are essentially bounded, $\| X_i \|$ are essentially bounded,

$$\left\| \frac{\partial X_i}{\partial x_j} \right\| \leq L_k$$

---- " ----

A non-autonomous vf is autonomous when it is constant wrt t

• **Caratheodory theorem**: X_t a non-aut vf, then

$$(\star)\ \begin{cases} \dot{q}(t) = X(q(t)) \\ q(t_0) = q_0 \in M \end{cases}$$

has a unique solution γ defined on an open interval I, that respects $(\star)$ for a.e. $t \in I$ and moreover

$(t, q_0) \mapsto \gamma(t, t_0, q_0)$ is loc. Lipschitz wrt to t and smooth wrt to q_0

X_t is complete if γ is defined on $I = \mathbb{R}$, $\forall t_0 \in \mathbb{R}$

$P^X_{t_1 t_2} = \gamma(t_2, t_1, -)$ flow of X

$$\begin{cases} P_{t,t} = \mathrm{id} \\ P_{t_2,t_3} \circ P_{t_1,t_2} = P_{t_1,t_3} \\ P_{t_1,t_2}^{-1} = P_{t_2,t_1} \quad \text{(implied by the two above)} \end{cases}$$

If $P_{t,s}$ family of diffeomorphisms of M, respects the three above, then its infinitesimal generator

$$X_t := \frac{d}{ds} P_{t,s}\Big|_{s=t}$$

is <u>autonomous</u> iff $P_{0,t} \circ P_{0,s} = P_{0,t+s}$

Differentials of smooth maps

$\varphi : M \to N$ smooth map between smooth manifolds

$d\varphi : TM \to TN$, $\boxed{d\varphi(v) = \frac{d}{dt}\varphi(e^{tv}) \in T_{\varphi(q)}N}$

$v \in T_qM$

__Observation__ : $d(\varphi \circ \psi) = d\varphi \circ d\psi$ (COVARIANT behavior)

$\varphi_* = d\varphi$

__Q__ : what happens when we work with vector fields ?

If $X \in Vec(M)$ $\quad (d\varphi X)(\varphi(q)) := d\varphi(X(q))$ $\quad$ $M \xrightarrow{\varphi} N$, $q \longmapsto \varphi(q)$

so if $P \in Diff(M)$

then $(dPX)(q) = (P_* X)(q) = dP(X(P^{-1}(q)))$ $\quad \forall q \in M$

or $\quad (dPX)_q = dP(X_{P^{-1}(q)})$

Vector fields and diffeomorphisms as operators

$q \in M$, then q can be seen as an operator

$C^\infty(M) \to \mathbb{R}$, $\quad a \mapsto a(q)$, we write $\hat{q}a = a(q)$

$X \in \mathrm{Vec}(M)$, $\quad a \mapsto Xa$, we can write $\hat{X}a = Xa$

$P \in \mathrm{Diff}(M)$, $\quad a \mapsto a \circ P$, we can write $\hat{P}a = a \circ P$

Examples: $a_t = \widehat{e^{tX}}(a)$, $\quad X(q)a = \hat{q}\hat{X}a$ sometimes we write

$\hat{q} \odot \hat{X}$

In this notation $\widehat{P_* X} = \hat{P}^{-1} \odot \hat{X} \odot \hat{P} = Ad_{P^{-1}}(X)$

$$P_* X(q) = P_* X(P^{-1}(q))$$

if $a \in C^\infty(M)$, then $(P_* X)a = (X(a \circ P)) \circ P^{-1}$

If an operator gXg^{-1} makes sense this is denoted by $Ad_g(X)$

Lie Bracket

$X, Y \in \mathrm{Vec}(M)$

$$[X,Y] = \frac{\partial}{\partial t}\Big|_{t=0} e^{-tX}_{*} Y = \frac{\partial^2}{\partial t \partial s}\Big|_{t=s=0} e^{-tX} \circ e^{sY} \circ e^{tX}$$

Lemma: $[X,Y] = XY - YX$

→ **Proof**: $a \in C^\infty(M)$ and let's consider $\widehat{\left(e^{-tX}_{*} Y\right)} a =$

$$= \widehat{e^{tX}}\left(Y\left(a \circ e^{-tX}\right)\right) = \widehat{e^{tX}}\left(\hat{Y}\left(a - tXa + O(t^2)\right)\right) =$$

$$= \widehat{e^{tX}}\left(Ya - tYXa + O(t^2)\right) =$$

$$= Ya - tYXa + O(t^2) + tXYa - t^2XYXa + O(t^2) =$$

$$= Ya + t\,\underbrace{(XY - YX)}\,a + O(t^2)$$

therefore this is $\left.\frac{d}{dt}\right|_{t=0} \left(e^{-tX}_* Y\right) a$ $\square$

Exercises: (1) find the coordinate expression of $[X,Y]$

(2) $P \in \mathrm{Diff}(M) \Rightarrow P_*[X,Y] = [P_*X, P_*Y]$ $\forall X, Y \in \mathrm{Vec}(M)$

- **Proposition**: $X, Y \in Vec(M) \Rightarrow$ the following are equivalent

(1) $[X, Y] = 0$

(2) $e^{tX} \circ e^{sY} = e^{sY} \circ e^{tX}$

$\longrightarrow$ **Proof**: $(1 \Rightarrow 2)$ $\quad [X, Y] = 0 \Rightarrow e^{-tX}_* Y = Y$

$$\phi_s = e^{-tX} \circ e^{sY} \circ e^{tX} \Rightarrow \frac{\partial}{\partial s}\phi_s = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} e^{-tX} e^{(s+\varepsilon)Y} e^{tX} =$$

$$= \underbrace{\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} e^{-tX} \circ e^{\varepsilon Y} \circ e^{tX}}_{\boxed{e^{-tX}_* Y}} \circ \underbrace{e^{-tX} \circ e^{sY} \circ e^{tX}}_{\phi_s} = Y \circ \phi_s$$

I just proved that $\phi_s = e^{s Y}$

$$e^{-tX} \circ e^{sY} \circ e^{tX} = e^{sY}$$

by uniqueness of the flow

—— // ——

$(2 \Rightarrow 1)$ $\quad e^{tX} \circ e^{sY} \circ e^{-tX} = e^{sY}$

if we derive both sides wrt s, then

we get $\quad e^{-tX}_{*} Y = Y$

$\square$

Exercise: introduce $\gamma(t) := e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}$

$\eta : \mathbb{R}_{\geq 0} \to M$

$\Rightarrow$ prove that $\eta(t) = \gamma(\sqrt{t})$ is C^1 in a neighborhood of 0 and $\eta'(0) = [X, Y]$

$\rightarrow$ Proof: consider $u(t,s) = e^{-tY} e^{-sX} e^{tY} e^{sX}$

$$\Rightarrow \frac{\partial u}{\partial t} = -Y + Ad_{e^{-sX}}(Y) \; ; \; \frac{\partial u}{\partial s} = Ad_{e^{-tY}}(-X) + X$$

$$\frac{\partial^2 u}{\partial t \partial s} = 0 \; ; \; \frac{\partial^2 u}{\partial t^2} = [X, Y] \; ; \; \frac{\partial^2 u}{\partial s^2} = [Y, -X] = [X, Y]$$

$$\eta(t^2) = u(t,t)$$

and by Taylor expansion

$$\widehat{\eta(t^2)}a = a \circ \eta(t^2) = a(q) + t^2[X,Y]a + o(t^3)$$

and this concludes

$\square$