SR geodesics

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Sub-Riemannian manifolds

Vector distributions

II **Def**: M smooth manifold, D vector distribution of rank $= m$, is a family $D_q \subset T_qM$ subspaces $\forall q \in M$

the distribution is smooth $\forall q$ $\exists$ $U \ni q$ neigh. st

$$D_q = \operatorname{Span}\{X_1(q), \dots, X_m(q)\} \text{ for some } X_1, \dots, X_m \in \operatorname{Vec}(U)$$

D is involutive if $[D, D] = \text{span}\{[X, Y] \mid X, Y \in D\} = D$

therefore if $X_1, \dots, X_m$ is a local base then

$\exists\, a_{ij}^k$ st $[X_i, X_j] = \sum a_{ij}^k X_k$

D is flat if $\forall q_0 \in M$ $\exists$ loc. diffeomorphism

$\psi : \underset{\substack{\psi \\ q_0}}{U} \to \mathbb{R}^n$ st $\psi_* D_q = \mathbb{R}^m \times \{0\}$ $\quad \forall q \in U$

Theorem (Frobenius)

A smooth distribution is involutive iff it is flat

→ **Proof**: FLAT ⇒ INVOLUTIVE

$$D_q = \varphi_*^{-1}(\mathbb{R}^m \times \{0\}) \Longrightarrow D_q = \mathrm{Span}\{X_1, \dots, X_m\} \text{ if } X_i = \varphi_*^{-1}\frac{\partial}{\partial x_i}$$

$$[X_i, X_j] = [\varphi_*^{-1}\partial_i, \varphi_*^{-1}\partial_j] = \varphi_*^{-1}[\partial_i, \partial_j] = 0$$

INVOLUTIVE ⇒ FLAT

$D_q = \mathrm{Span}\{X_1(q), \dots, X_m(q)\}$, we want $e_*^{tX_k}(D) = D$

Let's define $Y_i^k(t) = e^{tX_k}_* X_i$

$$\frac{d}{ds} e^{-sX}_* Y = [X, Y]$$

$$\Rightarrow \quad \dot{Y}_i^k(t) = e^{tX_k}_* [X_i, X_k]$$

$$= e^{tX_k}_* \Big(\sum_{i,j} a_{ik}^j X_j \Big)$$

$$= \sum_{i,j} a_{ik}^j(t)\, Y_j^k(t)$$

$$\boxed{a_{ik}^j(t) := a_{ik}^j \circ e^{-tX_k}}$$

we denote by $\Gamma^j(t)$ the solution of the ODE

$$\dot{\Gamma}(t) = \big(a_{ik}^j(t)\big)_{ik}\, \Gamma(t)$$

$$\Gamma^j(t) = \left(\gamma^j_{ik}(t)\right)_{i,k}$$

and we finally have $y^j_i(t) = \sum_k \gamma^j_{ik}(t) \cdot y^j_k(0)$

and this proves the claim

because $e^{tX_j}_* X_i = \sum_k \gamma^j_{ik}(t) X_k$ and therefore

the pushforward of every X_i is again a

lin. composition of the X_ks

$$T_q M = \operatorname{Span}\{\underbrace{X_1, \ldots, X_m}_{\text{distribution}}, \underbrace{Z_{m+1}, \ldots, Z_n}_{\text{basis completion}}\}$$

$$\psi(t_1, \ldots, t_n) = e^{t_1 X_1} \circ e^{t_2 X_2} \cdots \circ e^{t_m X_m} \circ e^{t_{m+1} Z_{m+1}} \circ \cdots \circ e^{t_n Z_n}$$

$$\mathbb{R}^n \longrightarrow M$$

$$\frac{\partial \psi}{\partial t_i} = e^{t_1 X_1}_{*} \, e^{t_2 X_2}_{*} \cdots e^{t_i X_i}_{*} (X_i)(\psi(t_1, \ldots, t_n)) \in D_{\psi(t_1, \ldots, t_n)}$$

$$\Rightarrow \quad D_q = \psi_* \operatorname{Span}\left\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_m}\right\} \qquad \square$$

Therefore D is involutive $\Leftrightarrow$ loc. there is a sub-manifold $S \subset M$ st $\forall q \in S$, $D_q = T_q S$

A Sub-Riemannian manifold is a smooth manifold M such that $\exists\, D \subset TM$ smooth vector distribution

(1) $\mathrm{Lie}(D_q) = T_qM \quad \forall q \in M$ (Hörmander's condition)

$$\mathrm{Lie}(D_q) = \mathrm{Span}\{X, [X,Y], [[X,Y],Z], \dots \mid X, Y, Z, \dots \in D\}$$

$r(q)$ step of the distribution is the minimal length of the Lie brackets such that $\mathrm{Span}\{\text{Lie bracket of length} \le r\}$
$= T_qM$

(2) There exists on D a smooth scalar product

$\langle X, Y \rangle_{SR}$ positively def $\forall X, Y \in D$

A SR structure is the data of a vector bundle U

a sur. linear map f such that

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr] & TM \arrow[d] \\ & M \end{tikzcd}$$

there exist a scalar product on U

For every SR manifold there exists a trivial SR structure of some dimension such that

$$\begin{tikzcd} M \times \mathbb{R}^k \arrow[r, "f"] \arrow[dr] & TM \arrow[d] \\ & M \end{tikzcd}$$

$Im(f)_q = D_q$

and the scalar product on D is induced by the scalar product on $M \times \mathbb{R}^k$

therefore every $X \in D$ is $X = \sum_{i=1}^{k} u_i X_i$

Admissible curves

A lipschitz curve $\gamma : [0, T] \longrightarrow M$ is **admissible** if $\dot\gamma(t) \in D_{\gamma(t)}$

Working with "controls", $\dot\gamma(t) = \sum_i u_i(t) X_i(\gamma(t))$

with $u_i(t)$ ess. bounded and measurable

$$\dot\gamma(t) = f(\gamma(t), u(t))$$

Length of an admissible curve

$$\ell(\gamma) = \int_0^T \|\dot{\gamma}(t)\|_{SR}\, dt$$

- **Lemma**: $\ell(\gamma)$ is invariant under Lipschitz reparametrization

→ **Proof**: $\varphi : [0,T'] \to [0,T]$, $\gamma_\varphi = \gamma \circ \varphi$

$\dot{\gamma}_\varphi = \dot{\gamma}(\varphi(q)) \cdot \dot{\varphi}(q)$ so again this is Lipschitz & admissible

$$\ell(\gamma_\varphi) = \int_0^{T'} \|\dot\gamma_\varphi\| = \int_0^{T'} \|\dot\gamma(\varphi(t))\| \cdot |\dot\varphi(t)|\, dt =$$

$$= \int_0^{T} \|\dot\gamma(s)\|\, ds = \ell(\gamma)$$

$\square$

- <u>Lemma</u> every adm. curve with $\ell(\gamma) < \infty$ is a Lip. reparametrization of an adm. curve parametrized by arclength, meaning $\|\dot\gamma\|_{SR} = 1$

→ **Proof**

$$\ell(t) := \int_0^t \|\dot\gamma(s)\|\, ds \quad , \qquad \ell(T) = \ell(\gamma)$$

$\ell(t)$ is Lip. & monotonic

$$\zeta : [0, \ell] \longrightarrow M \quad \text{st} \quad \zeta(z) = \gamma(t) \quad \text{if} \quad z = \ell(t)$$

$$|\gamma(t_2) - \gamma(t_1)|_{Eucl.} \leq \int_{t_1}^{t_2} |\dot\gamma(s)|_{Eucl.}\, ds \leq C \cdot \int_{t_1}^{t_2} \|\dot\gamma(s)\|_{SR}\, ds$$

$$\Rightarrow |\zeta(s_2) - \zeta(s_1)| = |\gamma(t_2) - \gamma(t_1)| \leq C \cdot |\ell(t_2) - \ell(t_1)| = C \cdot |s_2 - s_1|$$

Moreover $\gamma(t) = \zeta(\ell(t))$; $\underline{\dot\gamma(t) = \dot\zeta(\ell(t)) \cdot \dot\ell(t)}$

$\dot\ell(t) = \|\dot\gamma(t)\|_{SR}$ if it exists, $\|\dot\zeta\|_{SR} = 1$ at every point $\ell(t)$ where $\dot\ell(t)$ is defined

$C_\ell := \{ s : s = \ell(t), \dot\ell(t)$ exists and $\dot\ell(t) = 0 \}$

has dimension 0 (Exercise)

$\square$

Minimal control

differentiability

$\gamma : [0,T] \to M$ admissible curve, at every point we can write

$$\dot\gamma(t) = \sum_{i=1}^{k} u_i(t)\, X_i(\gamma(t))$$

$u^*(t) = (u^*_1(t), \dots, u^*_k(t))$ is the minimal control

if I choose X_is of SR-norm $= 1$ &

$|u(t)| = \sqrt{\sum_i u_i^2(t)}$ is minimal among the k-uples st $\sum u_i(t) X_i(\gamma(t)) = \dot\gamma(t)$

the "minimal control" is the right definition of "norm" on a SR-structure.

Meaning that

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr] & TM \arrow[d] \\ & M \end{tikzcd}$$

$$\|v\| := \min\{|u| : f(q,u) = v\}$$

Lemma: $u^*(t)$ is measurable and essentially bounded

$$\xrightarrow{\qquad\qquad\qquad\qquad \mathcal{U} \qquad\qquad\qquad\qquad}$$

Sub-Riemannian distance

$\forall q_0, q_1 \in M$ SR-mfd

$$d_{SR}(q_0, q_1) = \inf \Big\{ \ell(\gamma) : \gamma : [0,T] \to M \text{ admissible curve}$$
$$\text{st } \gamma(0) = q_0 \ \& \ \gamma(T) = q_1 \Big\}$$

Raschevskii-Chow theorem

M sub-Riemannian mfd, connected, then

(1) (M, d_{SR}) is a metric space

(2) the topology induced by d_{SR} is the mfd topology

In particular d_{SR} is continuous wrt the mfd topology

→ <u>Proof</u> d respects the triangle inequality and is symmetric ✓

Next I want $\forall \varepsilon > 0 \ \forall q_0 \in M \ \exists U \ni q_0$ neighborhood such that $U_{q_0} \subset B_{SR}(q_0, \varepsilon)$

<u>Observation</u> the property above $\Rightarrow$ finiteness of the distance because the subset $\{q_2 : d(q_0, q_2) < \infty\}$ must be open then by connectedness $d(q_0, q_2) < \infty$ $\forall q_0, q_2 \in M$

__Lemma__ $N \subset M$ is a submfd, $\mathcal{F} \subset \mathrm{Vec}(M)$ such that

$X(q) \in T_qN \quad \forall X \in \mathcal{F} \quad \forall q \in N$

Then $\mathrm{Lie}_q(\mathcal{F}) \subset T_qN \quad \forall q \in N$

→ __Proof__: $\begin{cases} \dot{q} = X(q) \\ q(0) = q_0 \end{cases}$ has a unique solution locally

therefore if $X|_N \in TN$ then

$\Rightarrow e^{tX}(q) \in N$ for t small enough

by def. of $[X,Y] = \frac{d}{dt} e^{-tX}_* Y$ if X, Y are tangent to N

then $[X,Y]$ is also tangent to N

$\Longrightarrow$ iterating $\mathrm{Lie}_p(\mathcal{F}) \subset T_qN \quad \forall q \in N$ $\square$

__Lemma__ if $\mathcal{F} = \mathrm{span}\{X_1, \ldots, X_m\}$, $\mathrm{Lie}_q(\mathcal{F}) = T_qM \quad \forall q \in M$

then $\forall q_0 \in M \quad \forall V \subset \mathbb{R}^n$ a neighborhood of $\underline{0} \in \mathbb{R}^n$

$\exists\, \bar{s} = (\bar{s}_1, \ldots, \bar{s}_n) \in V$ and n vector fields

$Y_1, \ldots, Y_n \in \mathcal{F}$ such that $\bar{s}$ is a regular point

for the map $\psi : \mathbb{R}^n \to M$, $(s_1, \ldots, s_n) \mapsto e^{s_n Y_n} \circ \cdots \circ e^{s_1 Y_1}(q_0)$

Rmk: if $\mathcal{J}_{q_0} \subsetneq T_{q_0}M$ then $\bar{s}$ can't be $(0, \dots, 0)$

because $\mathrm{Im}(d\psi_{\underline{0}}) = \mathrm{span}\{Y_1, \dots, Y_m\} \subset \mathcal{J}_{q_0}$

→ Proof (A) dim 1 I take $Y_1 \in \mathcal{J}$ st $Y_1(q_0) \neq 0$

$\Rightarrow$ for $|s_1|$ small enough $s_1 \longmapsto e^{s_1 X_1}$ is

a loc. diff. between $\mathbb{R}$ and $\Sigma_1 = \mathrm{Im}(e^{s_1 X_1}(q_0))$

$$\psi_1 : s_1 \longmapsto e^{s_1 X_1}(q_0)$$

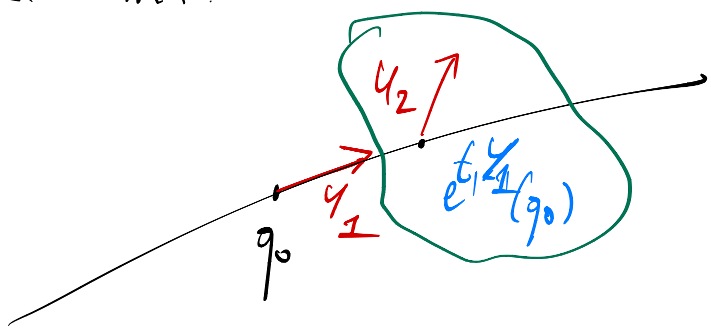
(B) dim $= 2$

$\exists\, t_1$ sufficiently small $\exists\, Y_2 \in \mathcal{F}$ such that

Y_2 is <u>not</u> tangent Σ_1 at $e^{t_1 X_1}(q_0)$

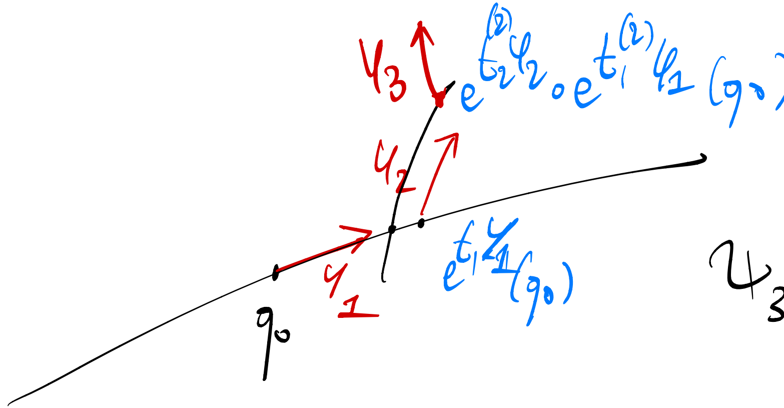
$$\psi_2 : (s_1, s_2) \longmapsto e^{s_2 Y_2} \circ e^{s_1 Y_1}(q_0)$$

this is a loc. diff.
around $(t_1, 0)$



(c) dim $= 3$

Then $\exists\ t_1^{(2)}, t_2^{(2)}$ suff. small such that at $\psi_2(t_1^{(2)}, t_2^{(2)})$, $\exists\ Y_3 \in \mathcal{Y}$ which is not tangent to $\Sigma_2 = \mathrm{Im}(d\psi_2)$

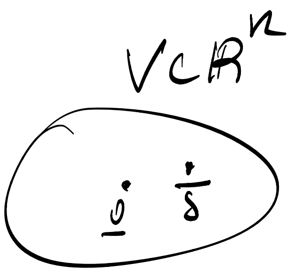


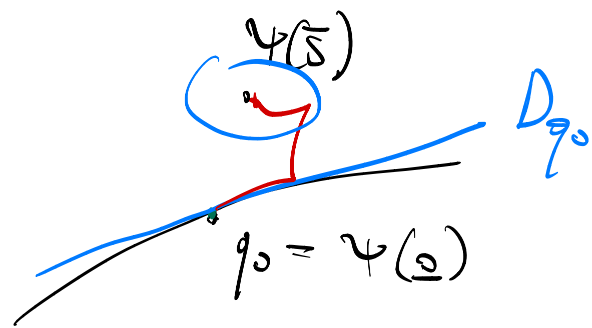
$$\psi_3(s_1, s_2, s_3) = e^{s_3 Y_3} \circ e^{s_2 Y_2} \circ e^{s_1 Y_1}(q_0)$$

After n steps we have

$$\psi : (s_1, \dots, s_n) \longmapsto e^{s_n Y_n} \circ e^{s_{n-1} Y_{n-1}} \circ \cdots \circ e^{s_1 Y_1}(q_0)$$

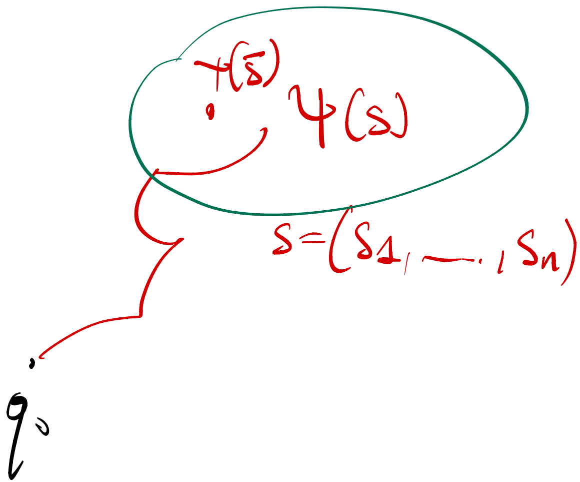
and is regular at $(\bar{s}_1, \dots, \bar{s}_n) = (t_1^{(n)}, t_2^{(n)}, \dots, 0)$

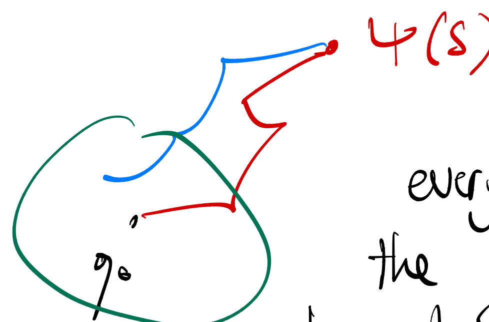




Observe that we can put such a system of loc. coordinates around q_0

$$(s_1, \dots, s_n) \mapsto e^{-\bar{s}_1 Y_1} \circ \dots \circ e^{-\bar{s}_n Y_n} \circ e^{s_n Y_n} \circ \dots \circ e^{s_1 Y_1}(q_0)$$





every point in the green neigh. has $d_{SR} < 2n \cdot \varepsilon$

We want to prove $\forall\, U \ni q_0$ neigh. $\exists\, \varepsilon > 0$

$B_{SR}(q_0, \varepsilon) \subset U$

Observe that if $K \subset M$ is a compact with $q_0 \in \mathrm{int}(K)$

$\exists$ local coordinates on K then $\exists\, \delta_K > 0$

such that $\forall\, \gamma$ admissible starting at q_0 at

$\ell(\gamma) \le \delta_K$ does not exit K

$$C = \max_{\substack{q \in K \\ \|v\|_{SR} = 1}} |v|_{Eucl} \qquad v \in D_q \subset T_q M$$

we choose δ_K such that $C \cdot \delta_K < d_{Eucl}(q_0, \partial K)$

$$\Rightarrow \ell_{Eucl}(\gamma) = \int_0^T |\dot\gamma(s)| \, ds \leq \int_0^T C \cdot \|\dot\gamma(s)\| \, ds \leq C \cdot \ell_{SR}(\gamma)$$