SR geodesics

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21/1/26

* Good behaviour of the length minimizers
* Recalling Hamiltonian formalism
* Pontryagin extremals
* Symplectic formalism

Good behaviour of the length minimizers

Raschevskii–Chow: (M, d_{SR}) is a metric space and the topology induced by d_{SR} is equivalent to the mfd topology

Corollary: (M, d_{SR}) is locally compact

→ **Proof:** by continuity $\overline{B_{SR}(q,r)}$ is closed $\exists\, r$ small enough st $\overline{B(q,r)} \subset U$ for every open neigh $U \ni q$
$\Rightarrow \overline{B(q,r)}$ is closed & bounded $\square$

A length minimizer is an adm. curve γ such that $\ell(\gamma) = d(\gamma(0), \gamma(1))$

- Thm: if $\gamma_n : [0,1] \to M$ adm. and constant speed ($\|\dot\gamma_n\|$ is constant) and $\gamma_n \to \gamma$ uniformly

if $\liminf \ell(\gamma_n) < +\infty \Rightarrow \gamma$ admissible

and $\ell(\gamma) \leq \liminf \ell(\gamma_n)$

→ Proof: up to a subseq. $\ell(\gamma_n) \to L = \liminf \ell(\gamma_n)$

$\forall q \in M$, $V_q = \{ X \in D_q : \|X\| \leq L + \delta \} \subset T_q M$

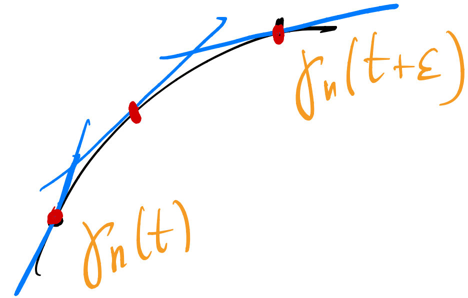
for n big enough $\dot{\gamma}_n(t) \in V_{\gamma_n(t)}$

We want to estimate

$$|\gamma_n(\tau) - \gamma(t)|_{\text{Eucl.}} \leq |\gamma_n(\tau) - \gamma_n(t)| + \underbrace{|\gamma_n(t) - \gamma(t)|}_{\text{behaves well by unif. conv.}}$$

where τ is sufficiently near to t

$$\gamma_n(\overbrace{t+\varepsilon}^{\tau}) - \gamma_n(t) = \int_t^{t+\varepsilon} \dot\gamma_n(s)\, ds$$



$$\in \mathrm{Conv}\left(V_{\gamma_n(s)} : s\in[t,t+\varepsilon] \right)$$

$$|\dot\gamma_n| = \frac{|\dot\gamma|}{\|\dot\gamma\|_{SR}} \cdot \|\dot\gamma\|_{SR}$$

$$|\gamma_n(t+\varepsilon) - \gamma_n(t)|_{Eucl.} \le \int_t^{t+\varepsilon} |\dot\gamma_n(s)|\, ds = \le C \cdot (L+\delta)\cdot \varepsilon$$

Therefore γ is Lipschitz (by putting this estimate in the previous slide ineq.)

$$\frac{\gamma(t+\varepsilon)-\gamma(t)}{\varepsilon} \in \mathrm{Conv}\left(V_q : q \in B(\gamma(t), r_\varepsilon) \right)$$

with $r_\varepsilon \to 0$ for $\varepsilon \to 0$

in the limit $\dot\gamma(t) \in \mathrm{Conv}(V_{\gamma(t)}) = V_{\gamma(t)}$

this is true for every $\delta \Rightarrow \|\dot\gamma\|_{SR} = L$

$\square$

Corollary if γ_n seq. of length minimizers of constant speed and $\gamma_n \to \gamma$ uniformly then γ is a length minimizer

Because by what we said

$$l(\gamma) \leq \liminf l(\gamma_n) = \liminf \left(d(\gamma_n(0), \gamma_n(1)) \right) = d(\gamma(0), \gamma(1))$$

$\square$

- <u>Thm</u>: if $\overline{B(q_0,r)}$ is compact, then $\forall p_1 \in B(q_0,r)$ $\exists$ length minimizer btw q_0 & q_1

<u>Observation</u>: M is loc. cpt $\Rightarrow$ a pair point near enough has always length minimizers btw them

Idea $\mathrm{Im}(\gamma_n) \subset \overline{B(q_0,r)}$ $\forall n$ therefore we get unif Lipschitzianity and therefore we can use A-A.

$\square$

Recalling some Hamiltonian formalism

Legendre transform for some f which is convex and C^1
(f' is invertible)

$$f^*(p) := \sup_x \left(px - f(x)\right)$$

if $x = x(p)$ is argsup then $f'(x) = p$

$$f^* = p \cdot x(p) - f(x(p)) \quad \text{and} \quad (f^*)'(p) = x(p)$$

therefore $f^{**} = f$

in Lagrangian formalism we want to minimize some action $A = \int_{t_0}^{t_1} L(q(t), \dot{q}(t))\, dt$

$\Rightarrow$ the well known relation $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$

so we look at $L(q, v)$ as a function in v and define $p := \frac{\partial L}{\partial v} \Rightarrow \dot{p} = \frac{\partial L}{\partial q}$

$$H = L^* = pv - L(q, v)$$

because of the involutive property

$$L(q,v) = pv - H(q,p)$$

with $p = p(q,v)$ and $v = \frac{\partial H}{\partial p}$

if we add the constraint $v = \dot{q}$

then we get the usual

hamiltonian system

$$\begin{cases} \dot{p} = -\dfrac{\partial H}{\partial q} \\ \dot{q} = \dfrac{\partial H}{\partial p} \end{cases}$$

Pontryagin extremals

We denote by γ_u the curve solving

$$\begin{cases} \dot{\gamma}_u(t) = \sum_i u_i(t) X_i(\gamma(t)) \\ \gamma_u(0) = q_0 \end{cases}$$

where $u_i(t)$ are some control functions in $L^\infty([0,1] \to \mathbb{R})$

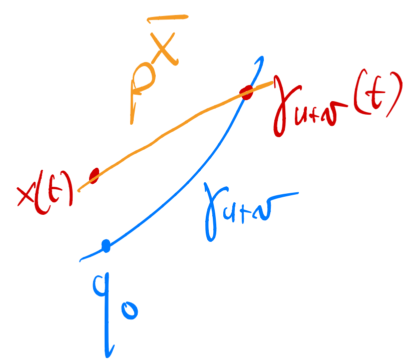
and $X_1, \ldots, X_m$ span the distribution

We are going to consider γ_{u+v} with $v \in L^\infty([0,1] \to \mathbb{R})$

$$\dot{\gamma}_{u+v}(t) = \sum \big(u_i(t) + v_i(t)\big) X_i(\gamma_{u+v}(t))$$

we define $x(t) = x_{u+v}(t)$ such that

$$\gamma_{u+v}(t) = P_t^{\overline{X}}(x(t)) \qquad \overline{X} := \sum u_i(t) X_i \quad \left(\begin{array}{l}\text{non-aut}\\ \text{vec. field}\end{array}\right)$$



$$\dot{\gamma}_{u+v}(t) = \sum_i u_i X_i(\gamma_{u+v}(t)) + v_i X_i(\gamma_{u+v}(t))$$

$$\frac{d}{dt} P_t^{\bar{X}}(x(t)) = \bar{X}\left(P_t^{\bar{X}}(x(t))\right) + \left(P_t^{\bar{X}}\right)_* \dot{x}(t)$$

But these are the same and $\sum u_i X_i = \bar{X}$

$$\Rightarrow \quad \dot{x}(t) = \left(P_t^{\bar{X}}\right)_*^{-1}\left(\sum v_i X_i\right)$$

$$= \sum_i v_i \tilde{X}_i \qquad \text{if} \quad \boxed{\tilde{X}_i = \left(P_{-t}^{\bar{X}}\right)_* X_i}$$

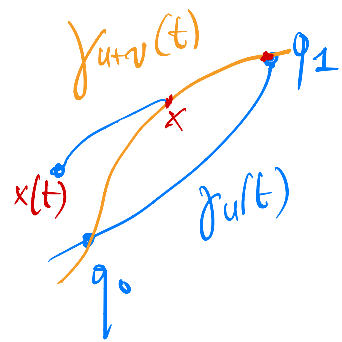
Now we plug inside all this the constraints that we have.

$\gamma_{u+v}(0) = q_0$ is coded in the definition

horizontality " " " " "

$$\gamma_{u+v}(1) = q_1$$

$$x(1) = \left(P_1^{\bar{X}}\right)^{-1}_{*}(q_1) = q_0$$



$J(u) : L^\infty \to \mathbb{R}$ $\qquad J(u) = \frac{1}{2}\int_0^1 \sum u_i^2(t)\, dt$

we are minimizing J instead of $\int_0^1 \sqrt{\sum_i u_i^2(t)}\, dt$

because in this way the minimizers are constant speed

We want to minimize J with the constraint

$x(1) = q_0$

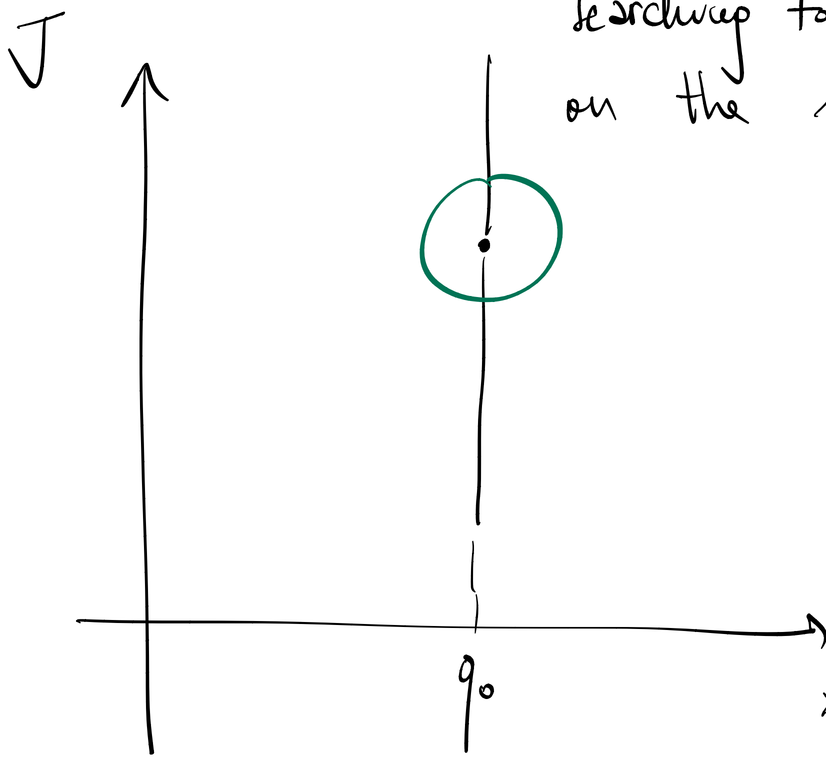
By the Lagrange multipliers technique

$$\exists\ \bar{\lambda} \in \left(\mathbb{R} \oplus T_{\varphi_0} M\right)^* = \mathbb{R} \oplus T^*_{\varphi_0} M \quad \text{st}$$

$$\left\langle \bar{\lambda} \, , \, \begin{matrix} \dfrac{\partial J(u+sv)}{\partial s} \\ \\ \dfrac{\partial x(1;u+sv)}{\partial s} \end{matrix} \right\rangle \Bigg|_{s=0} = 0$$

where we are looking at J and $x(1)$ as functions of v

searching for a degeneracy point of J on the line $x(1) = q_0$



if for every $\lambda \in (\mathbb{R} \oplus T_{q_0} M)^*$ the evaluation was $\neq 0$ then (dJ, dx) would be a loc. diff. at u and every point in the green neighb. would be reachable

$$J(u+s v) = \frac{1}{2}\int_0^1 \sum (u_i(t) + v_i(t))^2 \, dt$$

$$\Rightarrow \frac{\partial J}{\partial s} = \int_0^1 \sum u_i(t) \cdot v_i(t) \, dt$$

Moreover $\dot{x}(t; u+sv) = \sum s v_i \cdot \tilde{X}_i$

$$\Rightarrow \frac{\partial \dot{x}}{\partial s}(t; u+sv)\Big|_{s=0} = \sum v_i \tilde{X}_i(\underbrace{x(t;u)}_{})$$

is always go
therefore I can integrate it

We get $\displaystyle \frac{\partial x}{\partial s}(1; u+sv)\Big|_{s=0} = \int_0^1 \sum_i v_i(t) \cdot \tilde{X}_i(q_0)\, dt$

After re-normalization $\bar{\lambda} \in \mathbb{R} \oplus T^*M$

$$\bar{\lambda} = \begin{pmatrix} -1 \text{ or } 0 \\ \lambda_0 \end{pmatrix} \text{ then } \left\langle \bar{\lambda}, \begin{matrix} \frac{\partial J}{\partial s} \\ \frac{\partial x(1)}{\partial s} \end{matrix} \right\rangle = -\frac{\partial J}{\partial s} \text{ or } 0 + \left\langle \lambda_0, \frac{\partial x(1)}{\partial s} \right\rangle$$

$$\left\langle \lambda_0, \frac{\partial x}{\partial s}(1;u) \right\rangle = \left\langle \lambda_0, \int \sum v_i \cdot \tilde{X}_i(q_0)\, dt \right\rangle$$

$$\tilde{X}_i = (P^{\bar{x}}_{-t})_* X_i$$

$$= \int \left(\sum v_i \cdot \langle \lambda_0, \tilde{X}_i \rangle \right) dt$$

$$\lambda(t) = (P^{\bar{x}}_{-t})^* \lambda_0$$

$$= \int_0^1 \sum v_i(t) \cdot \underline{\langle \lambda(t), X_i \rangle}\, dt$$

$$\frac{\partial J}{\partial s} = \int_0^1 v_i \cdot \underline{u_i}\, dt$$

this must be true

$\forall$ choiche $v = (v_1, \ldots, v_m)$

in L^∞

In the case we treated this implies

$$u_i(t) = \langle \lambda(t), X_i \rangle$$

for a.e. $t \in [0,1]$

$i = 1, \dots, m$

NORMAL (PONTRYAGIN) EXTREMALS

$$0 = \langle \lambda(t), X_i \rangle$$

ABNORMAL EXTREMALS

the extremals live in the cotangent bundle
we are interested in the proj on M

Symplectic formalism

$\lambda(t) := \left(P_{-t}^{\overline{X}}\right)^* \lambda_0$ for some $\lambda_0 \in T^*_{q_0} M$

$\Longrightarrow \lambda(t) \in T^*_{\gamma_u(t)} M$

this is a curve in T^*M

$\left(P_{-t}^{\overline{X}}\right)^*$ is a family of diffeomorphisms in T^*M

we would like to know some v.f. $V_{\overline{X}} \in \mathrm{Vec}(T^*M)$ st

$$\left(P_{-t}^{\overline{X}}\right)^* = P_t^{V_{\overline{X}}} : T^*M \longrightarrow T^*M$$

Consider $\lambda \in T^*M$, $\lambda = (q, p)$ with $q \in M$, $p \in T_q^*M$

we want to work with

$V \in T_\lambda T^*M$

we know that v.f. are associated to derivations

$$C^\infty(T^*M) \longrightarrow C^\infty(T^*M)$$

$f \in C^\infty(T^*M)$, $f(q,p) = f(q,0) + df_q(p) + R(q,p)$ " $o(|p|)$

Taylor wrt p

We can study the action of derivations on $C^\infty(T^*M)$

$\pi : T^*M \to M$

by focusing on the action on

(i) fiber constant functions $\alpha(\lambda) = \alpha(\pi(\lambda))$

(ii) linear functions (on the fibers) $\lambda \longmapsto \langle \lambda, \varphi \rangle$

for some $\varphi \in \mathrm{Vec}(M)$

$h_\varphi(\lambda) = \langle \lambda, \varphi \rangle$

Suppose we have an autonomous vector field X

then for fiber constant functions

$$\frac{d}{dt}\alpha\left(\left(e^{-tX}\right)^*(\lambda_0)\right) = X\alpha(q_0)$$

$$\frac{d}{dt}h_Y\left(\left(e^{-tX}\right)^*(\lambda_0)\right) = \langle \lambda_0, [X,Y] \rangle = h_{[X,Y]}(\lambda_0)$$

these uniquely determine a derivation on $C^\infty(T^*M)$

we call this derivation V_X

$$V_X \alpha = X\alpha$$

$$V_X h_Y = h_{[X,Y]}$$

<u>Definition</u>: The Poisson bracket $\{\cdot,\cdot\}: C^\infty(T^*M)\times C^\infty(T^*M)\to C^\infty(T^*M)$ is the unique bilinear skew-simm form $\{a,b\}=-\{b,a\}$ such that $\{h_X, h_Y\} = h_{[X,Y]}$ and it is a derivation $\{a,bc\} = \{a,b\}c + \{a,c\}b$

For any $a \in C^\infty(T^*M)$ we use the notation $\vec{a}$ meaning the derivation $\{a, \cdot\}$

After some minor verification we get that

$$V_X = \{h_X, -\}$$

therefore $V_X = \vec{h}_X$

If (x,p) are the coordinates on T^*M $\quad x=(x_1,\dots,x_n)$, $p=(p_1,\dots,p_n)$

then $\quad a(x,p)=\alpha(x)$

$h_X(x,p)=\sum p_i v_i \qquad$ if $X=\sum v_i e_i$

(here $p_i = e_i^*$)

$$\{a,b\} = \sum_{i=1}^{n} \frac{\partial a}{\partial p_i}\frac{\partial b}{\partial x_i} - \frac{\partial a}{\partial x_i}\frac{\partial b}{\partial p_i}$$

$$\vec{a} = \sum_i \frac{\partial a}{\partial p_i}\frac{\partial}{\partial x_i} - \sum \frac{\partial a}{\partial x_i}\cdot\frac{\partial}{\partial p_i}$$

$$\vec{h_X} = \sum_i v_i \frac{\partial}{\partial x_i} - \sum_{i,j} p_i \cdot \frac{\partial v_i}{\partial x_j}\cdot\frac{\partial}{\partial p_j}$$

We showed that if X is autonomous

$$\left(P^X_{-t}\right)^* = \left(e^{-tX}\right)^* = e^{t\vec{h}_X} \quad \text{in } T^*M$$

but the same can be proven also if X_t is a non-aut vector field

$$\left(P^{X_t}_{-t}\right)^* = P_t^{\vec{h}_{X_t}}$$

Consider the 1-form $s \in \Omega^1(T^*M)$ such that

$$\langle s_\lambda, w \rangle := \langle \lambda, \pi_* w \rangle \quad \forall\, w \in T_\lambda(T^*M)$$

then in classic coordinates we have

$$s = \sum_i p_i \, dx_i$$

tautological or Liouville 1-form

$\sigma = ds = \sum_i dp_i \wedge dx_i$ is a closed non-deg. 2-form on T^*M. Is called canonical symplectic structure

By linear algebra

$$\{a,b\} = \vec{a}(b) = \langle db, \vec{a} \rangle = \sigma(\vec{a}, \vec{b})$$

we conclude here, in the next lesson we start by studying $\lambda(t) = e^{t\vec{h_{\bar{x}}}}(\lambda_0)$ with these formalisms