

SR geodesics

4

23/1/26

- * normal extremals
- * abnormal extremals

Examples

- * 2D-Riemannian geometry
- * Isoperimetric problem
- * Heisenberg group H^1 and other "contact" geometries

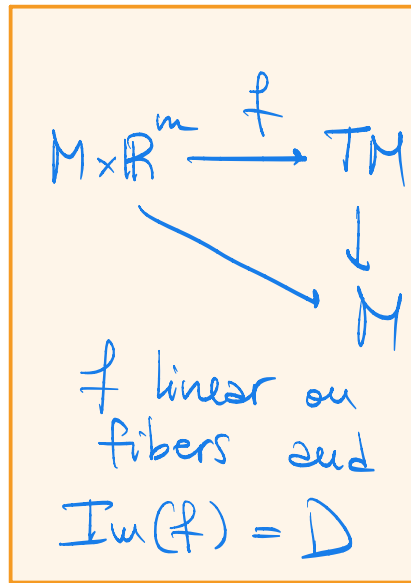
Recall

Every sub-Riemannian wfd is equivalent to a free sub-Riemannian structure

Therefore, $\exists$ a generating family

$X_1, \dots, X_m$ st $D = \text{span}\{X_1, \dots, X_m\}$

and $\|X\|_{SR} = \min_{\sum u_i X_i = X} \sqrt{\sum u_i^2}$



We saw that, if $\gamma_u(t)$

minimizes $J(u) = \frac{1}{2} \int_0^1 \left(\sum u_i^2(t) \right) dt$

with the constraint $\gamma_u(1) = q_1$, then

$$(1) \gamma_u(t) = \pi(\lambda(t)) \text{ for } \lambda(t) = (P_{t,0}^{\bar{X}})^* \lambda_0$$

$$\lambda_0 \in T_{q_0}^* M$$

$$(2) \begin{cases} (N) \langle \lambda(t), X_i \rangle = u_i(t) \\ (A) \langle \lambda(t), X_i \rangle = 0 \end{cases}$$

for all $t \in [0, 1]$

$\forall i = 1, \dots, m$

γ_u defined by

$$\begin{cases} \dot{\gamma}_u(t) = \sum u_i(t) X_i \\ \gamma_u(0) = q_0 \end{cases}$$

We use the notation

$$\bar{X} = \sum u_i(t) X_i$$

$$\pi: T^*M \rightarrow M$$

We also showed that $\lambda(t) = P_{0,t}^{\vec{h}_X}(\lambda_0)$
in the cotangent bundle T^*M

Recall:
 $\lambda = (x, p)$

This means that $\dot{\lambda}(t) = \vec{h}_X(\lambda(t)) = u_i(t) \vec{h}_i$

In coordinates if $\dot{\lambda}(t) = \vec{h}$

$$\begin{cases} \dot{p}_j = -\frac{\partial h}{\partial x_j} \\ \dot{x}_j = \frac{\partial h}{\partial p_j} \end{cases}$$

where $h_i(\lambda) = h_{x_i}(\lambda) = \langle \lambda, X_i \rangle$

$h = h(x, p)$ because
 $h \in C^\infty(T^*M)$

Also, by our formalism, if $a \in C^\infty(T^*M)$
and $a_t = a \circ e^{t\vec{H}}$

Then $\dot{a}_t = \{H, a\}$ $H \in C^\infty(T^*M)$

Normal extremals

$$h_i(\lambda(t)) = \langle \lambda(t), X_i \rangle = u_i(t) \quad \forall i=1, \dots, m$$

for a.e. $t \in [0, 1]$

$$\Rightarrow \dot{\lambda}(t) = \sum_{i=1}^m h_i(t) \overrightarrow{h_i}$$

By definition, $\sum_i h_i \overrightarrow{h_i} = \sum_i h_i \{h_i, -\} = \frac{1}{2} \sum_i \{h_i^2, -\}$

$$\Rightarrow \dot{\lambda}(t) = \overrightarrow{H} \quad \text{where} \quad H(\lambda) = \frac{1}{2} \sum_i \langle \lambda, X_i \rangle^2$$

we show that H does not depend on $\{X_1, \dots, X_m\}$

$$H(\lambda) = \frac{1}{2} \sum_i \langle \lambda, X_i \rangle^2 = \max_u \left(\langle \lambda, \sum u_i X_i \rangle - \frac{1}{2} \|u\|^2 \right)$$

$$= \max_{\substack{X \in D_q \\ q \in \pi(\lambda)}} \left(\langle \lambda, X \rangle - \frac{1}{2} \|X\|_{SR} \right)$$

if we have some u_i st $\sum u_i X_i = X$
 then the minimal control for X
 does the best result

Exercise: maximization problem

the argmax is $\sum_{i=1}^m \langle \lambda, X_i \rangle X_i$ ($u_i = \langle \lambda, X_i \rangle$ is 2
 minimal control)

• Theorem: $\lambda: [0,1] \rightarrow T^*M$ Lipschitz, is a normal extremal
 iff it is a solution of $\dot{\lambda}(t) = \vec{H}(\lambda(t))$

and moreover $\dot{\gamma}(t) = \pi(\dot{\lambda}(t))$

$$\Rightarrow \frac{1}{2} \|\dot{\gamma}(t)\|_{SR}^2 = H(\lambda(t))$$

→ Proof: $\dot{\lambda}(t) = \sum_{i=1}^m h_i \vec{h}_i = \sum_{i=1}^m \langle \lambda, X_i \rangle \vec{h}_i$

$$\dot{\gamma}(t) = \pi(\dot{\lambda}(t)) = \sum_{i=1}^m \langle \lambda, X_i \rangle X_i \quad \text{because } X_i = d\pi(\vec{h}_i)$$

□

In canonical coordinates

$$H(x, p) = \frac{1}{2} \sum \langle p, X_i \rangle^2$$

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} = \sum \langle p, X_i \rangle X_i \\ \dot{p} = -\frac{\partial H}{\partial x} = -\sum \langle p, X_i \rangle \cdot \langle p, D_x X_i \rangle \end{cases}$$

Observation: $\alpha \cdot \lambda(\alpha t) = (x(\alpha t), \alpha \cdot p(\alpha t))$ is still a solution $\approx$

$$\pi(\lambda(t\beta)) = \gamma(t\beta)$$

for some
 $\beta \in [0, 1]$

so the projection of the flow of
 α multiple of $\vec{H}$ has the same image

← π →

$$\gamma(t) = \pi(e^{t\vec{H}}(\lambda_0)) , \quad \|\dot{\gamma}\|_{SR} = H(\lambda_0)$$

therefore, curves $p.$ by arclength live in $H^{-1}(1/2)$

$\omega \in \Omega^2(M)$, γ is a characteristic curve

for ω if $\omega(\dot{\gamma}, -) \equiv 0$, $\dot{\gamma} \in \ker(\omega)$ for $t \in [0, 1]$

• Theorem: $\lambda: [0, 1] \rightarrow T^*M$ Lipschitz that stays

in $H^{-1}(c)$, with $c \in \mathbb{R}$ a regular value of H ,

then

λ is a (repar. of ω) $\iff$ λ is a characteristic curve for $\sigma|_{H^{-1}(c)}$

canonical symplectic 2-form

→ Proof: $E_\lambda := T_\lambda H^{-1}(c) \subset T_\lambda T^*M$, $\dim(M)=n$, $\dim(T^*M)=2n$

$$\dim(E_\lambda) = 2n - 1$$

Over M , $\ker(\sigma) = \{0\}$ therefore $\dim(\sigma_\lambda|_{E_\lambda}) = 1$

by definition $\sigma_\lambda(-, \vec{H}(\lambda)) \Big|_{E_\lambda} = d_\lambda H \Big|_{E_\lambda} = 0$

$$\Rightarrow \dot{\lambda}(t) = \varphi(t) \cdot \vec{H}$$

□

Abnormal extremals

$$\dot{\lambda}(t) = \sum \mu_i \vec{h}_i \quad ; \quad h_i(\lambda) = \langle \lambda, X_i \rangle = 0 \quad \forall i$$

$$\Leftrightarrow H(\lambda) = 0$$

Observe that in this case $d_\lambda H = \sum \langle \lambda, X_i \rangle X_i = 0$

but we have the constant rank theorem, if $\dim D_g = m$ is constant then $H^{-1}(0)$ is a smooth mfd

of $\text{codim} = m$

Theorem: M SR-wfd (D of constant rank),
 $\lambda \in H^{-1}(0)$ Lipschitz curve, then

λ is a (repr. of an) abnormal extremal $\iff \lambda$ is a characteristic curve of $\sigma|_{H^{-1}(0)}$

Proof: $N := H^{-1}(0) = \{\mu : h_i(\mu) = 0 \ \forall i\} \subset T^*M$

$$\begin{aligned} T_\lambda N &= \{\omega \in T_\lambda T^*M \mid \langle d_\lambda h_i, \omega \rangle = 0 \ \forall i\} \\ &= \{\omega \in T_\lambda T^*M \mid \sigma(\omega, \vec{h}_i) = 0 \ \forall i\} = \text{span}\{\vec{h}_i\} \xrightarrow{\perp} \sigma \end{aligned}$$

We proved $\text{Ker } \sigma_\lambda|_N = \text{Span} \{ \vec{h}_i \}$.

Therefore $\dot{\lambda}(t) = \sum u_i(t) \vec{h}_i$

with $h_i(\lambda) = 0$ $\square$

⚠ Everything in the "abnormal setting" depends on D but not on the metric

Example 1 : 2D-Riemannian geometry

M a Riemannian surface

Here no lon. extremals because $\langle \lambda, X_i \rangle = 0 \quad \forall i$
 $\Rightarrow \lambda = 0$

Let's consider a loc. orthonormal frame

$$X_1, X_2 \quad \dot{\gamma}(t) = u_1 X_1 + u_2 X_2$$

$$h_i(\lambda) = \langle \lambda, X_i \rangle$$

$$H(\lambda) = \frac{1}{2} (h_1^2 + h_2^2)$$

A normal extremal respects

$$\dot{\gamma}(t) = h_1(t) X_1 + h_2(t) X_2$$

$$\dot{h}_i = \frac{d}{dt}(h_i(\lambda(t))) = \overrightarrow{H}(h_i)$$

$$\dot{h}_1 = \{H, h_1\} = -\{h_1, h_2\} h_2$$

$$\dot{h}_2 = \{H, h_2\} = \{h_1, h_2\} h_1$$

H has to be constant, $H \equiv \frac{1}{2}$, $h_1^2 + h_2^2 = 1$

$$\{h_2, h_1\} = \dot{\theta}$$



$$\begin{array}{cc} \cos \theta & \sin \theta \\ | & / \end{array}$$

$$[X_1, X_2] = c_1 X_1 + c_2 X_2 \quad c_i \in C^\infty(M)$$

$$\{h_1, h_2\}(\lambda) = \langle \lambda, [X_1, X_2] \rangle = c_1 h_1 + c_2 h_2$$

$$(1) \quad \dot{\theta} = c_1(\gamma(t)) \cos \theta(t) + c_2(\gamma(t)) \sin \theta(t)$$

$$(2) \quad \dot{\gamma} = \cos \theta(t) X_1 + \sin \theta(t) X_2$$

every curve that satisfies (2) is pba, it is an extremal if it satisfies (1)

In fact $\dot{\theta} - c_1 \cos(\theta) - c_2 \sin(\theta)$ is the geod. curv. of $\gamma(t)$

Observe that if $\lambda(t) \in T^*M$ is pba

$$\lambda(t) \in S^*M \subset T^*M$$

$$\Rightarrow \lambda(t) = \cos \theta X_1 + \sin \theta X_2 + \dot{\theta} \partial_{\theta}$$

Isoperimetric problem

$$M = \mathbb{R}^2, \quad A \in \Omega^1(\mathbb{R}^2)$$

we search $\inf \{ l(\gamma) \mid \gamma(0) = q_0; \gamma(1) = q_1; \int_{\gamma} A = c \}$

we consider a SR setting in $\overline{M} = M \times \mathbb{R}$

$\Sigma = (q, z)$ coordinates on $\overline{M}$

given $\gamma(t) \subset M$, $\Sigma(t) = (\gamma(t), z(t) = \int_0^t A(\dot{\gamma}(s)) ds)$

$$\Rightarrow \text{if } \omega = dz - A, \quad \dot{\xi} \in \text{Ker}(\omega)$$

We consider $D = \text{Ker}(\omega)$ distribution on $\overline{M}$

$$D = \text{span} \{ F_i = X_i + \langle A, X_i \rangle \partial_z \mid i=1,2 \}$$

$$\dot{\xi} = u_1 F_1 + u_2 F_2 \quad \text{also we set } F_0 := \partial_z$$

$$\text{and } h_i := \langle -, F_i \rangle \quad i=0,1,2$$

$$\zeta(t) = \pi \left(e^{t\vec{H}}(\lambda_0) \right)$$

$$\vec{H} = \frac{1}{2}(h_1^2 + h_2^2) \quad , \quad h_1 = \cos \theta \quad , \quad h_2 = \sin \theta$$

$$H \equiv \frac{1}{2} \quad \quad \dot{h}_i = \{H, h_i\} \quad , \quad \dot{h}_0 = 0$$

$$[F_1, F_2] = c_1 F_1 + c_2 F_2 + \underbrace{dA(X_1, X_2)}_{\text{In the following,}} F_0$$

In the following,

$$b(q) := dA_q(X_1(q), X_2(q))$$

$$\{h_1, h_2\} = c_1 h_1 + c_2 h_2 + b \cdot h_0$$

the H. system becomes

$$\begin{cases} \dot{\zeta} = \cos \theta F_1 + \sin \theta F_2 \\ \dot{\theta} = c_1 \cos \theta + c_2 \cdot \sin \theta + b \cdot h_0 \\ \dot{h}_0 = 0 \end{cases}$$

$$\begin{aligned} \overline{M} &\xrightarrow{\pi_2} M \\ \gamma(t) &= \pi_2(\zeta(t)) \end{aligned}$$

$$k_g(\gamma(t)) = \dot{\theta} - c_1 \cdot \cos \theta - c_2 \cdot \sin \theta = b(f(t)) \cdot h_0(\gamma(t))$$

• Theorem non-constant abnormal extremals live in $b^{-1}(0)$

$$\rightarrow \text{Proof: } \dot{\lambda} = u_1 \overrightarrow{h_1} + u_2 \overrightarrow{h_2}, \quad h_1 \equiv h_2 \equiv 0$$

$$\Rightarrow \dot{h}_1 = 0 = \{\dot{\lambda}, h_1\} = u_2 \{h_2, h_1\}$$

$$\dot{h}_2 = 0 = \{\dot{\lambda}, h_2\} = -u_1 \{h_2, h_1\}$$

$$\text{but } (u_1, u_2) \neq 0$$

$$\Rightarrow \{h_1, h_2\} = 0 \Rightarrow (c_1 h_1 + c_2 h_2 + b \cdot h_0) \Big|_{\lambda(t)} = 0$$
$$\Rightarrow b|_{\lambda(t)} \equiv 0 \quad \square$$

Heisenberg group H^1

$$A = \frac{1}{2}(x dy - y dx)$$

$$X_1 = \partial_x$$

$$X_2 = \partial_y$$

$$dA = dx \wedge dy$$

$$F_1 = \partial_x - \frac{y}{2} \partial_z \quad ; \quad F_2 = \partial_y + \frac{x}{2} \partial_z$$

$$C_1 \equiv C_2 \equiv 0 \quad ; \quad b \equiv 1 \quad \left(\Rightarrow \text{no non-trivial} \right. \\ \left. \text{abn. geodesics} \right)$$

$$\begin{cases} \ddot{\gamma}(t) = \cos\theta \ F_1 + \sin\theta \ F_2 \\ \dot{\theta} = h_0 \\ \dot{h}_0 = 0 \end{cases}$$

(1) $h_0 \neq 0$

$$x(t) = \frac{1}{h_0} (\sin(\theta_0 + t h_0) - \sin(\theta_0))$$

$$y(t) = \frac{1}{h_0} (\cos(\theta_0 + t h_0) - \cos(\theta_0))$$

$$z(t) = \frac{1}{2} \int (\dot{x}y - \dot{y}x) dt = \frac{1}{2h_0^2} (t h_0 - \sin(t h_0))$$

(2) $h_0 = 0$

$$x(t) = \cos(\theta_0)t ; y(t) = \sin(\theta_0)t ; z(t) = 0$$

More generally if D distr. of codim 1,

$$\dim(D) = 2n-1$$

Suppose $D = \ker(\omega)$ for $\omega \in \Omega^1(M)$

Consider a basis $\omega_0 = \omega, \omega_1, \omega_2$ of T^*M

when $\dim(M) = 3$

$$D^\perp = \{ \lambda \in T^*M \mid \lambda(v) = 0, \forall v \in D \} = \langle \omega \rangle$$

$$\underbrace{s|_{D^\perp}}_{\text{Liouville form}} = \alpha_0 \cdot \omega \quad \alpha_0 \in C^\infty(M)$$

$$\sigma|_{D^\perp} = ds|_{D^\perp} = d\alpha_0 \wedge \omega + \alpha_0 \cdot d\omega$$

We search for char curves of $\sigma|_{H^{-1}(0)} = \sigma|_{D^\perp}$

$$\sigma|_{D^\perp} \text{ is degenerate iff } \sigma \wedge \sigma|_{D^\perp} = 0$$

because $\dim(D^\perp) = 4$

$$\sigma \wedge \sigma|_{D^\perp} = 2\alpha_0 \cdot (d\alpha_0 \wedge \omega \wedge d\omega)$$

let's check $\sigma \wedge \sigma|_{D^\perp}(\dot{\lambda}(t), \cdot, \cdot, \cdot)$

$$= 2 \alpha_0(\lambda(t)) \cdot (d\alpha_0(\dot{\lambda}(t)) \cdot (\omega \wedge d\omega(\cdot, \cdot, \cdot)) - \underbrace{\omega \wedge d\omega(\dot{\lambda}(t), \cdot, \cdot)}_{\text{this term is 0}} \cdot d\alpha_0(\cdot))$$

Conclusion: $\dim(M) = 3$, $D = \ker(\omega)$

then all non-trivial abn. extremals
are in the Martinet set

$$\mathcal{M} = \{ q \mid \omega \wedge d\omega = 0 \}$$

if $\omega \wedge d\omega \neq 0 \quad \forall q \Rightarrow$ the wfd is called
contact