Ordinary Differential Equations– IVP IV

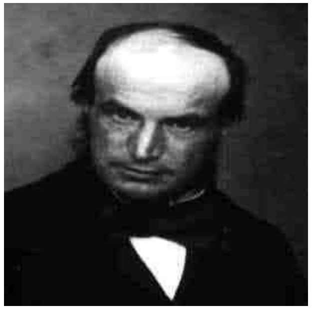
STIFFNESS

Numerical Methods for ODE

- **One-step Methods**
  - Euler’s Method
  - Analysis of the one-step methods
  - Runge-Kutta Methods
- **Multi-step Methods**
  - Adams-Bashforth
  - Adams-Moulton
  - Predictor-Corrector
- **Systems of ODE**
- **Stability**
- **Stiff Problems**

Leonhard Euler (1707-1783),

Martin Kutta
Carl David Runge (1856-1927)



J.C. Adams
(1819-1882)

Stiff Differential Equations

STIFF : Indicates a sort of ill-conditioning of the scalar or systems IVP that makes unstable almost all considered numerical methods.

An initial-value scalar/system of ODE is stiff if the step size needed to maintain absolute stability of the numerical method is much smaller than the step size needed to represent the solution accurately.

→ Stiff IVPs force explicit methods to use a very small step size and thus become unreasonably expensive.

For these problems we consider methods characterized by a "large" region of absolute stability

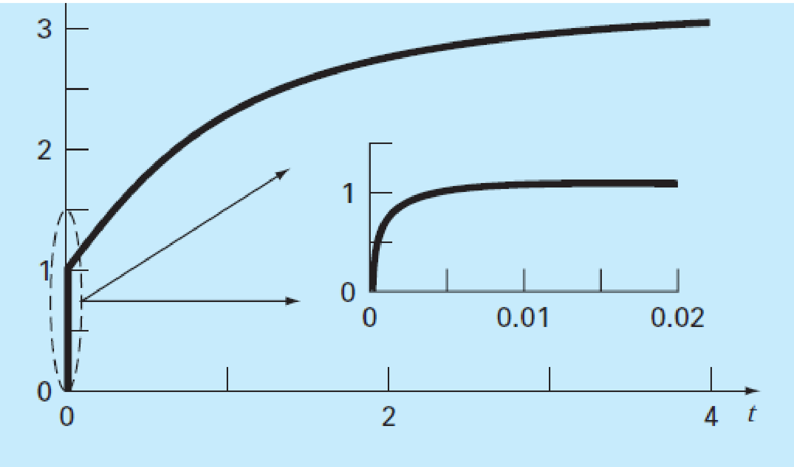
How to identify a stiff problem?

- Both scalar and systems of ODEs can be stiff.
- **Example:**

IVP:
$$y'(x) = -1000\,y(x) + 3000 - 2000e^{-x}$$
$$y(0) = 0$$

Exact sol: $$y(x) = 3 - 0.998e^{-1000x} - 2.002e^{-x}$$

although **the transient** occurs for only a small fraction of the integration interval, it controls the maximum allowable step size.



Explicit Euler's method

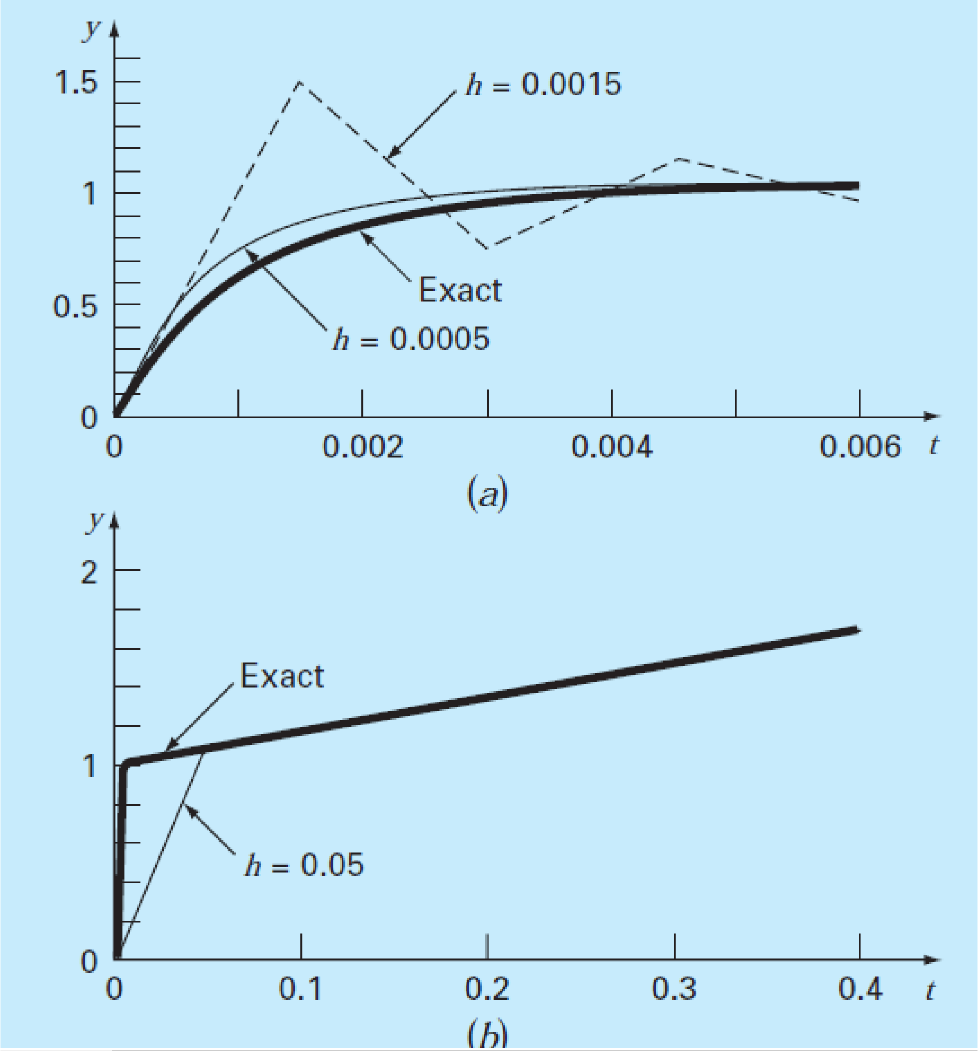
-stability limit h = 0.0015

-*for h* > 0.002

totally unstable solution

Implicit Euler's method

unconditionally stable.



How to identify a stiff problem?

- A system of linear ODE with constant coefficients

$$\boxed{y'(x) = Ay(x) + \phi(x) \qquad A \in \Re^{mxm},\ \phi \in \Re^{m}}$$

- Assume **A** has **m distinct eigenvalues**
- Solution

$$\boldsymbol{y(x) = \sum_{j=1}^{m} c_j e^{\lambda_j x} v_j + \psi(x)}$$

$c_1, c_2, \ldots, c_m$ constants

$\{v_j\}$ base formed by the eigenvectors of A $\lambda_j, j = 1,2,\ldots,m$

$\psi(x)$ particular solution

Example

- Problem 1

$$\begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} + \begin{pmatrix} 2\sin(x) \\ 2(\cos(x) - \sin(x)) \end{pmatrix}$$

$$\text{initial conditions} \quad \begin{pmatrix} y_1(0) = 2 \\ y_2(0) = 3 \end{pmatrix}$$

- Problem 2

$$\begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 998 & -999 \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} + \begin{pmatrix} 2\sin(x) \\ 999(\cos(x) - \sin(x)) \end{pmatrix}$$

$$\text{initial conditions} \quad \begin{pmatrix} y_1(0) = 2 \\ y_2(0) = 3 \end{pmatrix}$$

Same solution

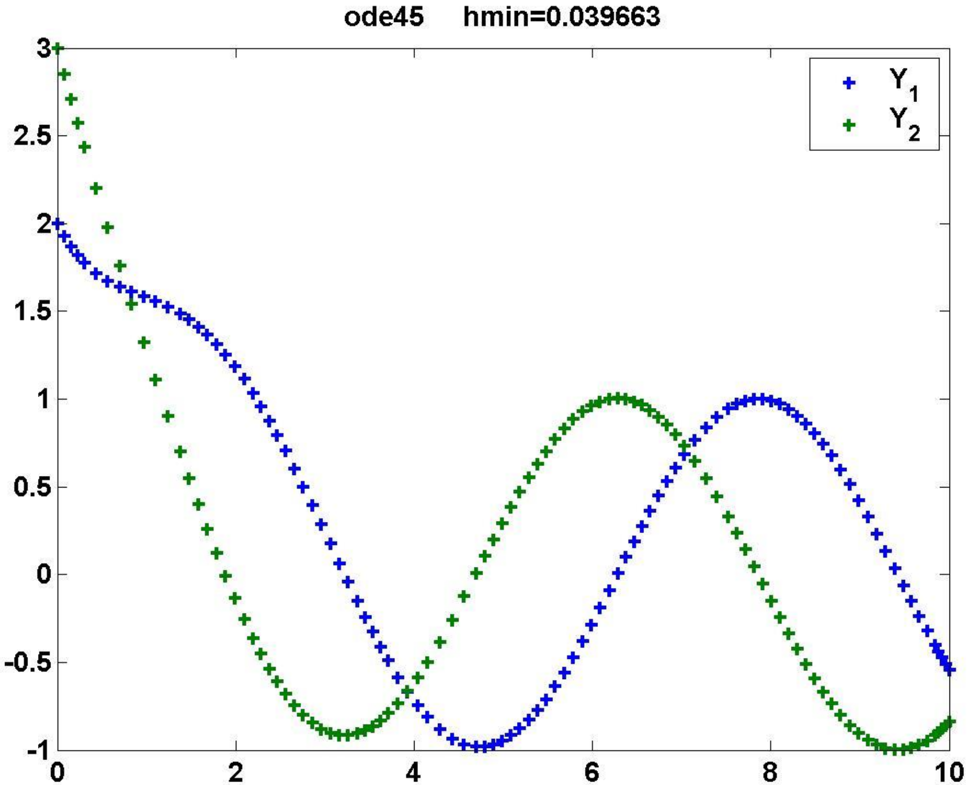
$$\boxed{\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = 2e^{-x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \sin(x) \\ \cos(x) \end{pmatrix}}$$

Numerical Solution

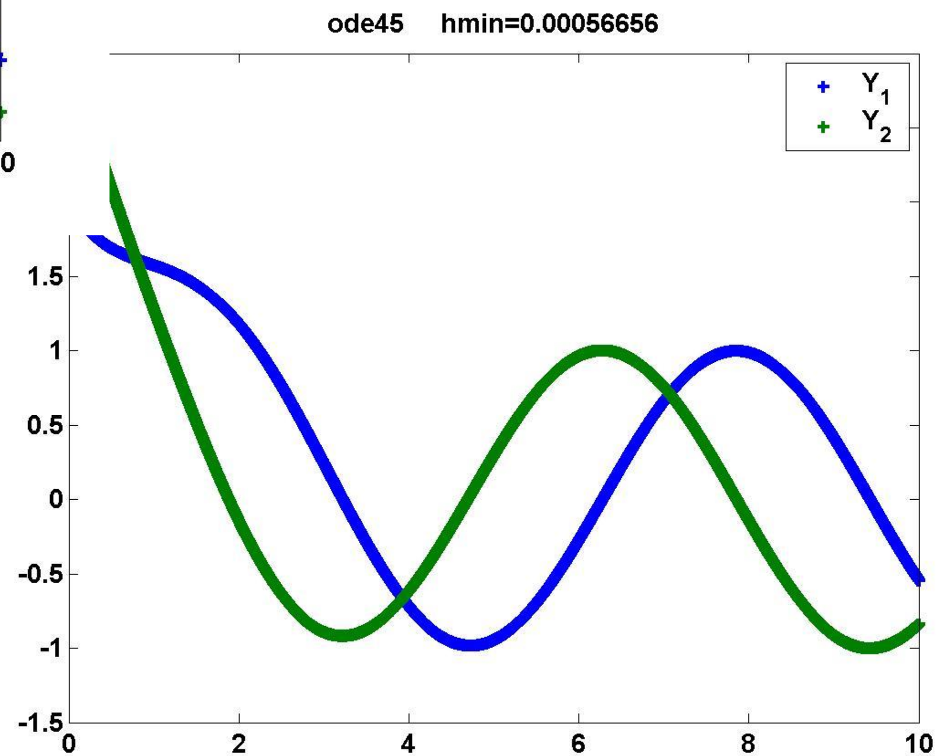
Problem 1	Problem 2
• RK Method of order 45 Adaptive step size with toll=0.01, in [0,10]. – **25** steps – **169** function eval.	• RK Method of order 45 Adaptive step size with toll=0.01, in [0,10]. - **3015** steps – **18769** function eval.
• Implicit Method of order 4 – **41** steps – **90** function eval.	• Implicit Method of order 4 – **48** steps – **112** function eval.

With fixed step h=0.1: the method fails (overflow)

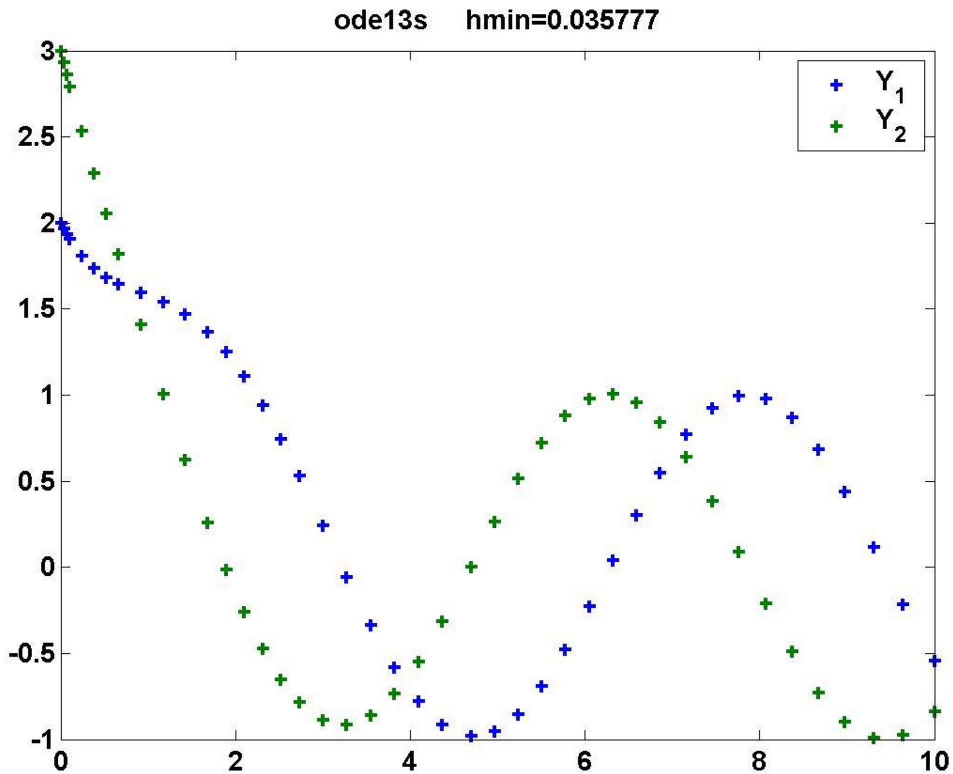
Problem 1

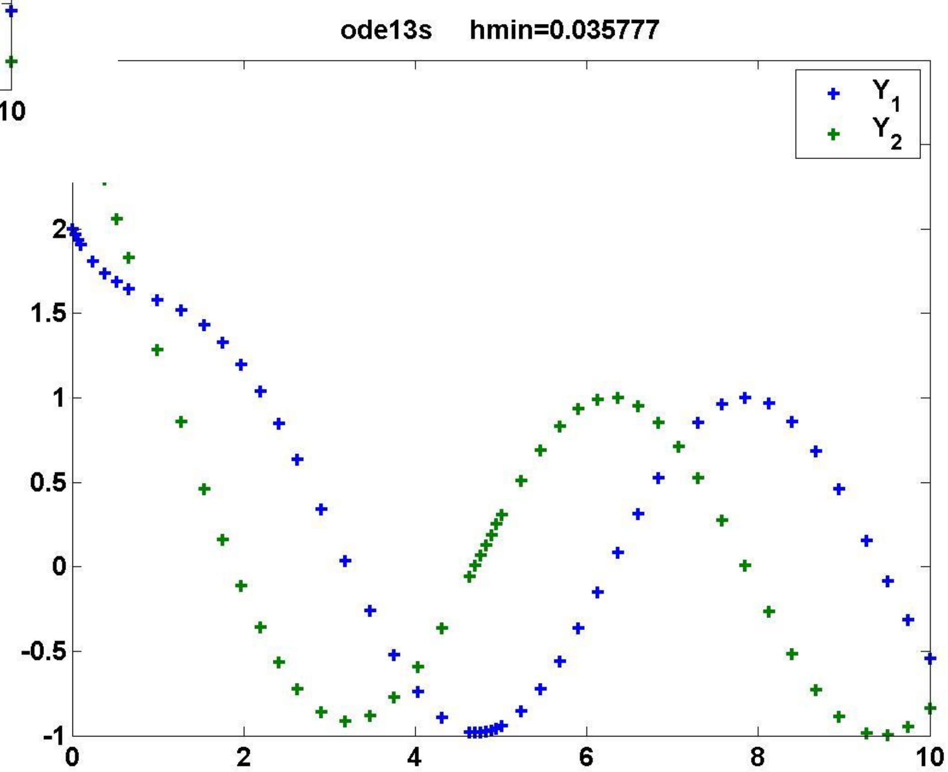


Problem 2



Problem 1





Problem 2

Remarks

- The problem is ill-conditioned, phenomenon known as stiffness
- Problem 2 is stiff,

  while problem 1 is not stiff.
- It's a property that does not depend on the solution, but on the problem data itself.

How to identify a stiff problem?

General solution

$$\boldsymbol{y(x)} = \sum_{j=1}^{m} \boldsymbol{c_j e^{\lambda_j x} v_j + \psi(x)}$$

In the example, the solutions can be written as

$$\boxed{\begin{aligned} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} &= c_1 e^{-x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3x} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} \sin(x) \\ \cos(x) \end{pmatrix} \\ \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} &= c_1 e^{-x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-1000x} \begin{pmatrix} 1 \\ -998 \end{pmatrix} + \begin{pmatrix} \sin(x) \\ \cos(x) \end{pmatrix} \end{aligned}}$$

- c_1 and c_2 are constants. Then for $x \to +\infty$ the y solution approaches to the particular solution Ψ, because each of the solutions $\boldsymbol{e^{\lambda_j x}}$ tends to zero for $x \to +\infty$
- Stability theory can give us a motivation..

PROBLEM 1 A eigenvalues

$$A=\begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix} \qquad \{\lambda=-1, \lambda=-3\}$$

PROBLEM 2 A eigenvalues

$$A=\begin{pmatrix}-2 & 1\\ 998 & -999\end{pmatrix} \qquad \{\lambda=-1, \lambda=-1000\}$$

Requirement for Absolute Stability: $h\lambda j$ be located in the method's absolute stability region Ra, for all eigenvalues λj of A.

Method RK 45 has absolute stability region (-3,0):

- for problem 1 $-3*h \in (-3,0)$ thus $h<1.0$
- for problem 2 $-1000*h \in (-3,0)$ thus $h<0.003$ and this is a strong limit for the time step h.

Method IMPLICIT 4 order has absolute stability region that includes the negative semi-axis of the complex plane, then $h\lambda \in Ra$ for each step h, when λ has real negative real part.

Characterization of stiffness for ODE systems

Solution: $$y(x) = \boxed{\sum_{j=1}^{m} c_j e^{\lambda_j x} v_j} + \psi(x)$$

Assume: $$\boxed{\mathrm{Re}\,\lambda_j < 0 \quad \forall j = 1,2,\dots,m.}$$

Then for $x \to +\infty$ the solution ***y*** tends to the particular solution Ψ , since each of the particular solutions $e^{\lambda_j x}$ tends to zero for ***x*** approaching infinity.

Ψ solution of the **steady-state** (i.e. for infinite times)

a solution of the **transient state** (for finite times)

Characterization of stiffness for ODE systems

- If $|\mathrm{Re}\,\lambda_j|$ is **large** it corresponds to a fast transient, a component of the solution (a) that decays very rapidly
- If $|\mathrm{Re}\,\lambda_j|$ is **small** it corresponds to a slow transient, a component of the solution that decays much more slowly.

Let

$$\mathrm{Re}\,\underline{\lambda} \le |\mathrm{Re}\,\lambda_j| \le \mathrm{Re}\,\overline{\lambda}$$

If we are interested to reach the steady state solution Ψ then we continue to integrate until the slower transient is not negligible. Smaller $|\mathrm{Re}\,\underline{\lambda}|$ and longer we will continue to integrate.

If we use a numerical scheme with region of absolute stability ($h^*\lambda \in$ Ra) and $|\mathrm{Re}\,\overline{\lambda}|$ is larger, then the step h must be very small for a very long period of time in order to obtain the stationary solution.

Then the step h appears to have limitations that depend on the maximum modulus of the eigenvalues of A.

Characterization of stiffness for ODE systems

We have stiffness when…

$\left|\mathrm{Re}\,\underline{\lambda}\right|$ is very small

$\left|\mathrm{Re}\,\overline{\lambda}\right|$ is very large

Stiffness Ratio

$$r_s = \frac{\left|\mathrm{Re}\,\overline{\lambda}\right|}{\left|\mathrm{Re}\,\underline{\lambda}\right|}$$

A system of linear ODE with constant coefficients is stiff if the eigenvalues of the matrix A all have negative real part and $r_s >> 1$

Numerical Methods for stiff ODE problems

- Problems for which A has eigenvalues with significantly different magnitudes are called stiff differential equations.

- No conditionally stable method is suitable for approximating a stiff problem.
- For such stiff problems, **implicit methods** (which generally have much larger stability regions) are generally favored. However, Implicit methods are more expensive.

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