Project 28 PDE
ADI method for two-dimensional diffusion equations with application to image denoising

1. INTRODUCTION TO THE PROBLEM

Consider the asymmetric two-dimensional equation

$$u_t = \nu u_{xx} + \sigma u_{yy}$$

with $\nu > 0$, $\sigma > 0$.
Solve the problem with the method of alternate implicit directions (ADI) where in each direction the Crank-Nicolson method is used for temporal integration.

Apply your model to the solution of the image denoising problem.
Consider as a computational domain the square [0,1] x [0,1] with zero Dirichlet boundary conditions. As initial data, choose a square image represented in grayscale on a white background and assign it to the function

$$u(x_i, y_j, 0) = (g_{max} - g_{ij}) / g_{max}, \quad i,j = 1,\dots,N$$

where $x_i = (i-1)/(N-1)$, $y_i = (j-1)/(N-1)$, N is the dimension of the matrix characterizing the image, gij the corresponding gray level and gmax the number total gray levels (corresponding to white). Illustrate the effect of applying the differential model on the image for different choices of ν and σ as the final time varies.
Add this new method to the software provided in the exercises for the image denoising problem and compare the results with the other implemented PDE-based methods.

2. DESCRIPTION OF THE MATHEMATICAL MODEL

The Alternating Direction Implicit (ADI) method is a method that was proposed by Douglas and Rachford in 1956, widely used for the numerical resolution of multidimensional parabolic PDEs.

In the two-dimensional Crank-Nicolson method

$$(I - \frac{r}{2}L)U^{n+1} = (I + \frac{r}{2}L)U^n$$

To solve 2-D parabolic PDEs with tridiagonal systems instead of using this pentadiagonal matrix, while maintaining the properties of the Crank-Nicolson method (unconditionally stable method for any time and space step), the ADI method is used. The matrix L can be written in two dimensions such as

$$L = L_x + L_y$$

where L_x represents the tridiagonal matrix which discretizes second derivative in x and L_y represents the tridiagonal matrix which discretizes second derivative along y.
Then the correct form of Crank-Nicolson becomes

$$(I - \frac{r}{2} L_x - \frac{r}{2} L_y) U^{n+1} = (I + \frac{r}{2} L_x + \frac{r}{2} L_y) U^n$$

The ADI method solves a simplified form of the Crank-Nicolson formula in the form (1) by separating the part in x from the part in y and then we can rewrite (1) as

$$(I - \frac{r}{2} L_x) U^{j+1/2} = (I + \frac{r}{2} L_y) U^j$$

$$(I - \frac{r}{2} L_y) U^{j+1} = (I + \frac{r}{2} L_x) U^{j+1/2}$$

where $U^{j+1/2}$ is a temporary vector variable.
Therefore, two linear systems are solved, the first with a tridiagonal matrix concerning the derivatives in x and the second with a tridiagonal matrix in y. The solution of the first linear system (temporary U) becomes the right-hand side of the second system and Thomas's algorithm is used to solve this mechanism.
So doing this splitting, which however is not the same as solving Crank-Nicolson, we obtain the resolution of two-dimensional PDEs using tridiagonal matrices, rather than pentadiagonal matrices. Clearly, it must be taken into account that the ADI method, by making approximations, has a lower accuracy than Crank-Nicolson.