Roberto Pagaria
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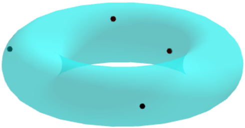
Unordered configuration spaces of surfaces

at UM topology seminar



March 24, 2022

Covered topics:

Let X be a topological space. Define:

$$\mathrm{F}_n(X) := \{(p_1, \ldots, p_n) \in X^n \mid p_i \neq p_j\}$$

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Example

$\mathrm{F}_n(\mathbb{R}^2)$ is the complement of the hyperplane arrangement of type A_{n-1}.

Delete a point

Theorem (Fadell, Neuwirth 1962)

If M is a manifold without boundary, then $p\colon \mathrm{F}_n(M) \to \mathrm{F}_{n-1}(M)$ is a fibration with fibre $M \setminus \{n-1 \text{ points}\}$.

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Recall the long exact sequence of homotopy groups:

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Corollary (Fadell, Neuwirth 1962)

If S is a surface different from S^2 and $\mathbb{P}_2(\mathbb{R})$, then $\mathrm{F}_n(S)$ and $\mathrm{C}_n(S)$ are $K(\pi, 1)$.

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Let M be a topological manifolds with boundary ∂M. The natural inclusion $\mathrm{F}_n(M \setminus \partial M) \to \mathrm{F}_n(M)$ is a homotopy equivalence.

Add a point

Theorem (Fadell, Neuwirth 1962)

If M is a non-compact manifold without boundary then the fibration $p\colon \mathsf{F}_n(M) \to \mathsf{F}_{n-1}(M)$ has a section.

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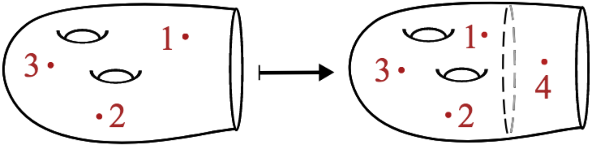

Theorem (Ellenberg, Wiltshire-Gordon 2015)

If M is a manifold that admits a non-zero vector field then $\dim H^i(\mathrm{F}_n(M);\mathbb{Q})$ *is polynomial in n.*
Moreover, for any $k > 0$ there exists a replication map $r\colon \mathrm{C}_n(M) \to \mathrm{C}_{kn}(M)$ *that induces isomorphism in lower degree in rational cohomology.*

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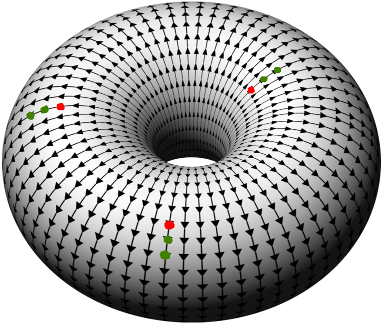

Closed manifolds

Example

The sphere S^2 does not admit isomorphisms in (co-)homology in lower degree, because
$H_1(\mathrm{C}_n(S^2);\mathbb{Z}) = H^2(\mathrm{C}_n(S^2);\mathbb{Z}) = \mathbb{Z}/(2n-2)\mathbb{Z}$.

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However, the obvious multivalued map $p\colon C_{n+1}(M) \rightrightarrows C_n(M)$ induces isomorphism in rational cohomology:

Theorem (Church 2011)

The map $p_*\colon H_i(C_{n+1}(M);\mathbb{Q}) \to H_i(C_n(M);\mathbb{Q})$ is an isomorphisms for $i < n$.

Remark

The condition $n > i$ is necessary since $H^2(\mathrm{C}_1(S^2); \mathbb{Q}) = \mathbb{Q}$ and $H^2(\mathrm{C}_n(S^2); \mathbb{Q}) = 0$ for $n > 1$.

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Let $i\colon N \hookrightarrow M$ be an inclusion of manifolds of the same dimension.

Theorem (Church 2011)

For each $k \leq n$, the map $i_*\colon H_k(\mathrm{C}_n(N);\mathbb{Q}) \to H_k(\mathrm{C}_n(M);\mathbb{Q})$ has constant rank (independent from n).

The Euler characteristic

Theorem (Felix, Thomas 2000)

Let M be an even-dimensional manifold. Then

$$\sum_{n=0}^{\infty} \chi(\mathrm{C}_n(M))u^n = (1+u)^{\chi(M)}$$

Moreover, $\chi(\mathrm{F}_n(M)) = n!\chi(\mathrm{C}_n(M))$.

The Betti numbers

Theorem (Drummond-Cole, Knudsen 2017)

Explicit calculation of the Betti numbers (i.e. $b_i(X) = \dim H^i(X)$) of $\mathrm{C}_n(S)$ for all surfaces S using the Chevalley-Eilenberg complex.

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For $4 < i < n$, the number $b_i(\mathrm{C}_n(\Sigma_g))$ is

$$-\binom{2g+i-1}{2g} - \binom{2g+i-4}{2g-1} + \sum_{j=0}^{g-1}\sum_{m=0}^{j}(-1)^{g+j+1}\frac{2j-2m+2}{2j-m+2}\cdot$$

$$\left[\binom{\frac{6j+2i+2g-2m+3-3(-1)^{i+j+g+m}}{4}}{m,2j-m+1} + \binom{\frac{6j+2i+2g-2m+1+3(-1)^{i+j+g+m}}{4}}{m,2j-m+1} + \right.$$

$$\left.\binom{\frac{6j+2i+2g-2m-3+3(-1)^{i+j+g+m}}{4}}{m,2j-m+1} + \binom{\frac{6j+2i+2g-2m-5-3(-1)^{i+j+g+m}}{4}}{m,2j-m+1}\right]$$

Differential graded algebras

Definition

A differential graded-commutative algebra (E, d) is a graded algebra $E = \oplus_{n\in\mathbb{N}} E^n$ and $xy = (-1)^{|x||y|} yx$ with a differential $\mathrm{d}\colon E \to E$ that satisfies the Leibniz rule $\mathrm{d}(xy) = \mathrm{d}(x)y + (-1)^{|x|} x\,\mathrm{d}(y)$.

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Example (A Koszul resolution)

Let V a finite dimensional vector space. The map $\mathrm{d}\colon \Lambda^\bullet V \otimes \mathrm{S}^\bullet V \to \Lambda^\bullet V \otimes \mathrm{S}^\bullet V$ defined by $\mathrm{d}(v \otimes 1) = 0$ and $\mathrm{d}(1 \otimes v) = v \otimes 1$ defines a dga.

Moreover, $H^i(\Lambda^\bullet V \otimes \mathrm{S}^\bullet V, \mathrm{d}) = 0$ for $i > 0$.

The Kriz model

Theorem (Kriz 1994)

Let M be a smooth projective variety. There exists a dga $(E(M), \mathrm{d})$ such that $H^\bullet(E(M), \mathrm{d}) \simeq H^\bullet(\mathrm{F}_n(M); \mathbb{Q})$.

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Let E be the exterior algebra on generators

- x_i for x in a basis of $H^\bullet(M)$ and $i \leq n$ with degree $(\deg x, 0)$,
- $G_{i,j}$ for $i < j$ with degree $(0, d-1)$,

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and relations

- $(x_i - x_j)G_{i,j} = 0$,
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The differential of degree $(d, 1-d)$ is given by

- $\mathrm{d}(x_i) = 0$,
- $\mathrm{d}(G_{i,j}) = [\Delta]_{i,j}$.

The group $\mathfrak{S}_n \times Sp(2g)$ acts on the Kriz model $E(\Sigma_g)$ by

$$(\sigma \times M) \cdot x_i = (Mx)_{\sigma(i)}$$

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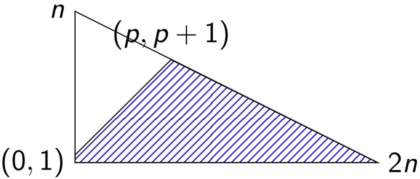
Theorem (Félix, Tanré 2005)

Let M be a even dimensional closed manifold, there exist an explicit dga $(C_n(M), \mathrm{d})$ such that:

$$(C_n(M), \mathrm{d}) \simeq (E(M), \mathrm{d})^{\mathfrak{S}_n}.$$

Moreover $C_n^r(M) = \Lambda^{n-2r} H(M) \otimes \Lambda^r sH(M)$, where s is the suspension.



Let $V = H^1(\Sigma_g)$, define the tri-graded algebra
$A_g = {\Lambda^\bullet[a,b]}/{(b^2)} \otimes \Lambda^\bullet V \otimes S^\bullet V$ with the differential $\mathrm{d}(b) = 0$, $\mathrm{d}(a) = \eta \in \Lambda^2 V$ and $\mathrm{d}(1 \otimes v) = b \otimes (v \otimes 1)$.

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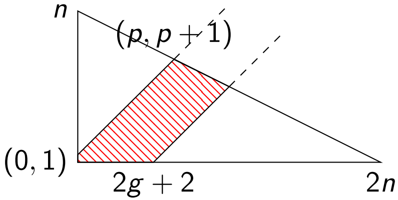
Theorem (P. 2019)

There exists an isomorphism

$$H(A_g^{\bullet,\bullet,\leq n}, \mathrm{d}) \simeq H^{\bullet,\bullet}(\mathrm{C}_n(\Sigma_g)).$$



Representation theory of $\mathfrak{sp}(2g)$

From the Lie theory the irreducible representations of $\mathfrak{sp}(2g)$ are parametrized by dominant weights, i.e. are isomorphic to V_λ for some vector $\lambda = a_1\omega_1 + a_2\omega_2 + \cdots + a_g\omega_g$, $a_i \in \mathbb{N}$.

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Let $\rho = g\omega_1 + (g-1)\omega_2 + \cdots + 2\omega_{g-1} + \omega_g$ be the half sum of positive roots.

Theorem (Weyl dimension formula)

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Example

$$\dim V_{i\omega_1 + \omega_j} = \binom{2g+i+1}{i,j} \frac{2g+2-2j}{2g+2+i-j} \frac{j}{i+j}$$

The algebra $\Lambda^\bullet V$

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The module $\Lambda^\bullet V$ as $Sp(2g)$-representation splits as:

$$
\begin{array}{ccccccccc}
0 & 1 & 2 & \dots & g & \dots & 2g-2 & 2g-1 & 2g \\
V_0 & & V_0 & \dots & V_0 & \dots & V_0 & & V_0 \\
 & V_{\omega_1} & & \dots & & \dots & & V_{\omega_1} & \\
 & & V_{\omega_2} & \dots & V_{\omega_2} & \dots & V_{\omega_2} & & \\
 & & & \ddots & \vdots & \iddots & & & \\
 & & & & V_{\omega_g} & & & &
\end{array}
$$

The multiplication by η moves "two on the right".

The dga $(\Lambda^\bullet V \otimes S^\bullet V, \mathrm{d})$

We need to compute $\ker \mathrm{d}$: in degree (j, i) it is isomorphic to $W_{i\omega_1+\omega_j}$ as representation of $\mathfrak{sl}(2g)$.

Theorem (Branching rule)

For $j \leq g$,

$$W_{i\omega_1+\omega_j} = \bigoplus_{0\leq 2k<j} V_{i\omega_1+\omega_{j-2k}} \oplus \bigoplus_{0\leq 2k<j-1} V_{(i-1)\omega_1+\omega_{j-2k-1}},$$

and $W_{i\omega_1+\omega_j} = W_{i\omega_1+\omega_{2g-j}}$ as representation of $\mathfrak{sp}(2g)$.

Mixed Hodge Theory

Let X be an algebraic variety, possibly non-projective and singular.

Theorem (Deligne 1974)

There exists a increasing filtration W_k of $H^i(X;\mathbb{Q})$ such that

$$\operatorname{gr}_k H^i(X;\mathbb{Q}) := W_k/W_{k-1}$$

admits a pure Hodge Structure of weight k.

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Example

The cohomology of the model (A_g, d) in degree (p,q) contributes to $\operatorname{gr}_{p+2q} H^{p+q}(\mathrm{C}(\Sigma_g))$.

The representation ring

The representation ring of a group G is $R(G)$, the $\mathbb{Z}$-module generated by all finite-dimensional representations V and relations

$$[V] + [W] = [V \oplus W].$$

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Example

$\dim\colon R(G) \to \mathbb{Z}$ is a morphism of ring.

Let

$$P_g(t,s,u) = \sum_{i,n,k} [\mathrm{gr}^W_{i+2k} H^{i+k}(\mathrm{C}_n(\Sigma_g))] t^i s^k u^n$$

in the representation ring $R(Sp(2g))[[t,s,u]]$.

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Theorem (P. 2019)

The series P_g is

$$\frac{1}{1-u}\Big((1+t^2su^3)(1+t^2u)+(1+t^2su^2)t^{2g}su^{2(g+1)}+(1+t^2su^2)\cdot$$
$$\cdot\,(1+t^2su^3)\sum_{\substack{1\leq j\leq g\\ i\geq 0}} [\mathbb{V}_{i\omega_1+\omega_j}] t^{j+i}s^i u^{j+2i}(1+t^{2(g-j)}su^{2(g-j+1)})\Big).$$

Thanks for listening!

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