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Hodge theory for polymatroids

joint work with Gian Marco Pezzoli

Arrangements at Home IV: Geometry and Topology

July 2, 2021

Covered topics:

Matroids

A *matroid* is an object that codifies the combinatorics of:

1. hyperplanes arrangements,
2. cycles of a graph,
3. linear dependencies among vectors.

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1. hyperplanes arrangements,
2. cycles of a graph,
3. linear dependencies among vectors.

There are a lot of equivalent definition:

1. rank function,
2. bases, independent sets, circuits,
3. geometric lattices,
4. integral polytopes, permutahedra, etc...

Subspace arrangements

Definition

A *subspace arrangement* in a complex vector space V is a finite collection of linear subspaces S_i of V.

Sometimes is useful to work with the projective version: the collection of $\mathbb{P}(S_i) \subset \mathbb{P}(V)$.

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For $I \subseteq [n] = \{1, 2, \dots, n\}$ define $\mathrm{cd}(I) = \mathrm{codim}_V(\cap_{i\in I} S_i)$ as the complex codimension of the *flat* $\cap_{i\in I} S_i$.

Example

In $\mathbb{C}^5$ consider S_a, S_b two subspace of dimension three and a line S_c in general position. We have $\mathrm{cd}(a) = 2, \mathrm{cd}(c) = 4$ and $\mathrm{cd}(ac) = \mathrm{cd}(abc) = 5$.

Polymatroids

A *polymatroid* P is a function $\mathrm{cd}\colon \mathcal{P}([n]) \to \mathbb{N}$ such that

1. $\mathrm{cd}(\emptyset) = 0$,
2. cd is increasing: $A \subset B$ implies $\mathrm{cd}(A) \leq \mathrm{cd}(B)$.
3. cd is submodular: $\mathrm{cd}(A) + \mathrm{cd}(B) \geq \mathrm{cd}(A \cap B) + \mathrm{cd}(A \cup B)$ for all A, B.

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There are equivalent definition in term of independent sets, bases, generalized permutahedra.
These objects codify the combinatorics of:

1. subspace arrangements,
2. cycles in an hypergraph.

A *flat* $F \subseteq [n]$ of codimension k is a maximal subset such that $\mathrm{cd}(F) = k$.

The poset of flats

Definition (Poset of flats)

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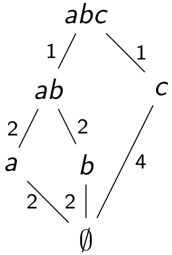
Example



In general L is not a geometric lattice and is not ranked.

Wonderful model

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(simple normal crossing divisor: the irreducible components are smooth and intersect locally as coordinate hyperplanes)
Let $\mathcal{G} \subset L$ be a "well chosen" collection of flats and consider

$$M \hookrightarrow \mathbb{P}(V) \times \bigtimes_{W \in \mathcal{G}} \mathbb{P}(V/W).$$

Let $Y_{\mathcal{G}}$ be the closure of the image of M.

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Theorem (De Concini, Procesi '95)

The variety $Y_{\mathcal{G}}$ is a wonderful model for M.

Building sets

A subset $\mathcal{G}$ of L is a *building set* if for all $x \in L$

$$[\hat{0}, x] = \prod_{y \in \max(\mathcal{G}_{\leq x})} [\hat{0}, y]$$

and

$$\mathrm{cd}(x) = \sum_{y \in \max(\mathcal{G}_{\leq x})} \mathrm{cd}(y).$$

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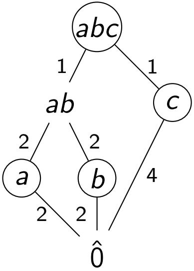

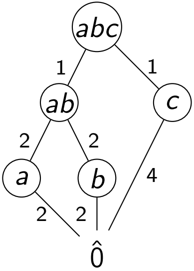

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A set $S \subseteq \mathcal{G}$ is *$\mathcal{G}$-nested* if the intersection $\cap_{W\in S} D_W$ is non-empty.
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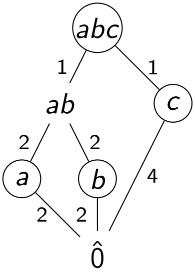

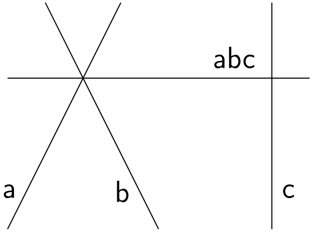

Previous works

- De Concini, Procesi '95 described the Chow ring $A(Y_{\mathcal{G}})$ (cohomology) of wonderful models.
- Feichtner, Yuzvinsky '03 described the Chow ring $A(L)$ of an atomic lattice with a building set.
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- Huh, Adiprasito, Katz '18 proved the Kähler package for $A(L)$ of a geometric lattice with the maximal building set.
- De Concini, Procesi '95 described the Leray model $B(\mathcal{G})$ for $M \hookrightarrow Y_{\mathcal{G}}$.
- Yuzvinsky '02, '99 simplified the model of De Concini Procesi and relate it to the Goresky-MacPherson formula.
- Bibby, Denham, Feichtner '21 studied the Leray model $B(\mathcal{G})$ for geometric lattices and partial building sets.

Leray model and Chow ring

The *Leray model* $(B^{\cdot,\cdot}(\mathcal{G}), \mathrm{d})$ is the second page of the Leray spectral sequence for $M \hookrightarrow Y_{\mathcal{G}}$ (aka the Morgan algebra).
Furthermore, $B^{\cdot,0}(\mathcal{G}) = H^{\cdot}(Y_{\mathcal{G}}) = A^{\cdot}(Y_{\mathcal{G}})$ and
$H^{\cdot}(B(\mathcal{G}), \mathrm{d}) = H^{\cdot}(M)$.

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Explicitly, $B^{\cdot,\cdot}(\mathcal{G})$ is generated by e_W, x_W for $W \in \mathcal{G}$ with bidegree $(0, 1)$ and $(2, 0)$ respectively and relations:

- $e_T x_S (\sum_{Z \geq W} x_Z)^b = 0$ for $S, T \subset \mathcal{G}$, $W \in \mathcal{G}$ and $b = \mathrm{cd}(W) - \mathrm{cd}(\bigvee(T \cup S)_{<W})$,

with differential defined by $\mathrm{d}(e_W) = x_W$.

(we use the notation $e_T = \prod_{W \in T} e_W$.)

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Explicitly, $A^{\cdot}(\mathcal{G})$ is generated by x_W for $W \in \mathcal{G}$ of degree 1 and relations:

- $x_S(\sum_{Z \geq W} x_Z)^b = 0$ for $S \subset \mathcal{G}$, $W \in \mathcal{G}$ and $b = \mathrm{cd}(W) - \mathrm{cd}(\bigvee(S_{<W}))$.

In the realizable case $x_W = [D_W]$ is the fundamental class of the (exceptional) divisor associated to W.

A second presentation

Define $\sigma_W = \sum_{Z \geq W} x_Z$ and $\tau_W = \sum_{Z \geq W} e_Z$. Geometrically, $\sigma_W \in A^1(Y_{\mathcal{G}})$ is the fundamental class of the total transform of W:

$$\sigma_W = [\pi^{-1}(W)],$$

where $\pi \colon Y_{\mathcal{G}} \to \mathbb{P}(V)$ is the canonical projection.

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The Leray model $B^{\cdot,\cdot}(\mathcal{G})$ is generated by τ_W, σ_W for $W \in \mathcal{G}$ with bidegree $(0,1)$ and $(2,0)$ respectively and relations:

- $\prod_{t \in T}(\tau_t - \tau_W) \prod_{t \in S}(\sigma_t - \sigma_W)\sigma_W^b = 0$ for $S, T \subset \mathcal{G}$, $W \in \mathcal{G}$ and $b = \mathrm{cd}(W) - \mathrm{cd}(\bigvee(T \cup S)_{<W})$,

with differential defined by $\mathrm{d}(\tau_W) = \sigma_W$.

Goresky MacPherson formula

Consider a subspace arrangement with complement M and poset of flats L.

Theorem (Goresky MacPherson '88)

There is an additive isomorphism

$$\tilde{H}^k(M;\mathbb{Z}) \cong \bigoplus_{W \in L \setminus \hat{0}} \tilde{H}_{2\,\mathrm{cd}(W)-2-k}(\Delta((\hat{0}, W)); \mathbb{Z}),$$

where $\Delta((\hat{0}, W))$ is the order complex of the interval $(\hat{0}, W)$.

We used the convention that $\tilde{H}_{-1}(\emptyset, \mathbb{Z}) = \mathbb{Z}$.

The critical monomial algebra

Theorem (Yuzvinsky '99, P. Pezzoli '21)

There exists a critical monomial algebra $\mathrm{CM}(\mathcal{G}) \subset B(\mathcal{G})$ *such that the inclusion is a quasi-isomorphism.*

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$$H^\bullet(\mathrm{CM}(\mathcal{G}), \mathrm{d}) \cong \bigoplus_{W \in L \setminus \hat{0}} \bigotimes_{Z \in \max(\mathcal{G}_{\leq W})} \tilde{H}_{2\,\mathrm{cd}(Z)-2-\bullet}(n(\mathcal{G}, Z)),$$

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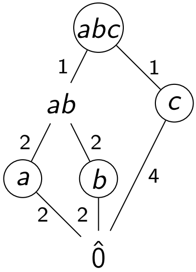
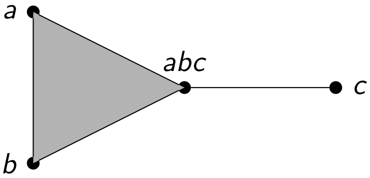
where $n(\mathcal{G}, Z)$ *is the simplicial complex of* $\mathcal{G}$*-nested sets in* $(\hat{0}, Z)$*.*





Definitions

Let A be an algebra with top degree n and $\deg\colon A^n \to \mathbb{Q}$ an isomorphism.

- the algebra A satisfies *Poincaré duality* if the bilinear pairing
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- the element $\ell \in A^1$ satisfies the *Hodge Riemann relations* if
$$Q_\ell^k\colon A^k \times A^k \to \mathbb{Q}$$
defined by $Q_\ell^k(a, b) = (-1)^k \deg(a\ell^{n-2k} b)$ (for $k \leq \frac{n}{2}$) is positive defined on the subspace
$$P_k = \ker(\cdot \ell^{n-2k+1}\colon A^k \to A^{n-k+1}).$$

Let L be a geometric lattice with $\mathrm{cd} = \mathrm{rk}$ and $\mathcal{G}$ be the maximal building set. The algebra $A(\mathcal{G})$ is the Chow ring of the matroid.

Theorem (Adiprasito, Huh, Katz '18)

The ring $A(\mathcal{G})$ is a Poincaré duality algebra and each $\ell = \sum_{W \neq \hat{1}} c_W x_W \in A^1(\mathcal{G})$ (ample) such that

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The Hodge Riemann relations prove a conjecture by Read, Hoggar, Rota, Heron, Welsh '60s-'70s:

Corollary (Adiprasito, Huh, Katz '18)

The coefficients of the characteristic polynomial for a log-concave sequence.

Let L be the poset of flats of a polymatroid and $\mathcal{G}$ an arbitrary building set.

Theorem (P. Pezzoli '21)

The ring $A(\mathcal{G})$ *is a Poincaré duality algebra and each* $\ell = -\sum_{W \in \mathcal{G}} d_W \sigma_W \in A^1(\mathcal{G})$ *such that*

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We call this orthant the σ-cone.

Remark

The σ-cone is contained in the ample cone of any realization, but for polymatroids the ample cone depends on the chosen realization.

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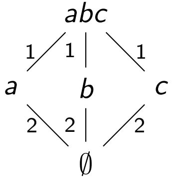
Example

Consider the polymatroid realized by three distinct lines in $\mathbb{C}^3$.



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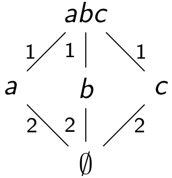

$Y_{\mathcal{G}}$ is the blowup of $\mathbb{P}^2$ in three points. If the three points are in general position then the ample cone coincides with the σ-cone.Otherwise the three points are collinear and the ample cone is given by:

$$\{-d_{abc}\sigma_{abc} - d_a\sigma_a - d_b\sigma_b - d_c\sigma_c \mid d_a, d_b, d_c > 0, \\ d_{abc} > -\min(d_a, d_b, d_c)\}$$

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There are examples of polymatroids with (reduced) characteristic polynomial with negative coefficients and that do not form a log-concave sequence.

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Remark

The main problem is that $x_{\hat{1}}$ behaves different from x_W for $W \in \mathcal{G} \setminus \hat{1}$.

Main lemmas

We needed to compute $\mathrm{Ann}(x_W)$ and $\mathrm{Ann}(\sigma_W)$:

Lemma

For $W \neq \hat{1}$ there is an isomorphism

$$A(\mathcal{G})\big/_{\mathrm{Ann}(x_W)} \cong A(\mathcal{G}_W) \otimes A(\mathcal{G}^W).$$

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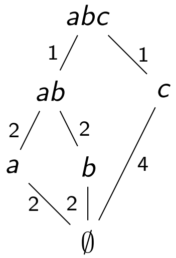
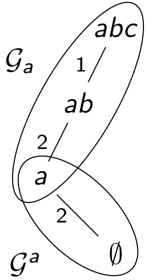
This looks like a Deletion-Restriction argument





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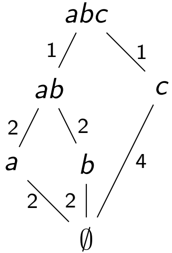
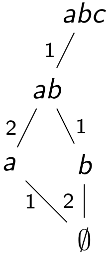
Idea: truncation at a consists in cutting the subspace arrangement with a generic hyperplane containing the flat a.





$\mathrm{tr}_a\,\mathcal{G}$

Sketch of the proof

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The Chow ring of a polymatroid satisfies the Kähler package.

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Sketch of the proof:

1. Present a Gröbner basis for $A(\mathcal{G})$,
2. Prove Poincaré duality constructing an explicit pairing,
3. Prove the previous lemmas using Poincaré duality,
4. Prove simultaneously Hard Lefschetz and Hodge Riemann by induction on $|\mathcal{G}|$.

Thanks for listening!

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