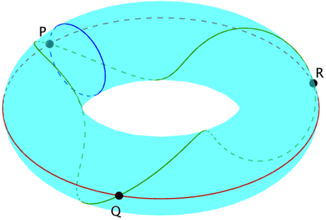
Roberto Pagaria
Scuola Normale Superiore

Orlik-Solomon-type presentations for the cohomology algebra of toric arrangements

Tropicalisation of Toric Arrangement Complements and Matroids over Rings



at Heilbronn Institute for Mathematical Research
September 5, 2019

Definitions

A *toric arrangement* $\mathcal{A}$ in the torus $T \simeq (\mathbb{C}^*)^n$ is a finite collection of (translates of) hypertori $\{ D_e \}_{e \in E}$. Let $\Lambda \simeq \mathbb{Z}^n$ be the character group of T and $\chi_e \in \Lambda$ a character defining D_e.
In coordinates: the characters are $\chi(t_1, \dots, t_n) = t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$ and the hypertori are

$$D = \{ (t_1, \dots, t_n) \in (\mathbb{C}^*)^n \mid t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n} = b \}$$

The equations $\chi(\boldsymbol{t}) = b$ and $(-\chi)(\boldsymbol{t}) = b^{-1}$ define the same hypertorus.

Definitions

A *toric arrangement* $\mathcal{A}$ in the torus $T \simeq (\mathbb{C}^*)^n$ is a finite collection of (translates of) hypertori $\{ D_e \}_{e \in E}$. Let $\Lambda \simeq \mathbb{Z}^n$ be the character group of T and $\chi_e \in \Lambda$ a character defining D_e.
In coordinates: the characters are $\chi(t_1, \dots, t_n) = t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$ and the hypertori are

$$D = \{ (t_1, \dots, t_n) \in (\mathbb{C}^*)^n \mid t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n} = b \}$$

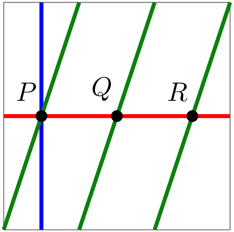
The equations $\chi(\boldsymbol{t}) = b$ and $(-\chi)(\boldsymbol{t}) = b^{-1}$ define the same hypertorus.

Definition

We say that $I \subset E$ is (in)dependent if the characters $\{\chi_e\}_{e \in I} \subset \Lambda \simeq \mathbb{Z}^n$ are linearly (in)dependent.

We want to study $M(\mathcal{A}) = T \setminus \bigcup_{e \in E} D_e$.

Example



Analogously to the case of hyperplane arrangements, we define

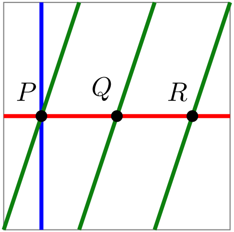
$$\omega_e = (b_e - \chi_e)^* \omega = -\frac{\mathsf{d}(t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n})}{b_e - t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}}.$$

Observe that

$$\omega_1 \cdot \omega_2 = \omega_{P;1,2} + \omega_{Q;1,2} + \omega_{R;1,2};$$

these two-forms are linearly independent.

Example



Analogously to the case of hyperplane arrangements, we define

$$\omega_e = (b_e - \chi_e)^* \omega = -\frac{\mathrm{d}(t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n})}{b_e - t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}}.$$

Observe that

$$\omega_1 \cdot \omega_2 = \omega_{P;1,2} + \omega_{Q;1,2} + \omega_{R;1,2};$$

these two-forms are linearly independent.

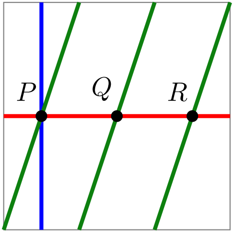
Choose $f_P = \frac{x^2+x+1}{3}$ and define the form:

$$\omega_{P;1,2} := f_P \cdot \omega_1 \cdot \omega_2 = \frac{x^2+x+1}{3} \mathrm{dlog}(1-y) \cdot \mathrm{dlog}(1-x^3 y)$$

The form $\omega_{P;1,2}$ depends on f_P, choosing $\tilde{f}_P = \frac{1}{3y}(x^2+x+1)$:

$$\tilde{\omega}_{P;1,2} := \tilde{f}_P \cdot \omega_1 \cdot \omega_2 = \omega_{P;1,2} + \omega_2 \cdot \mathrm{dlog}\, x.$$

Example



Analogously to the case of hyperplane arrangements, we define

$$\omega_e = (b_e - \chi_e)^* \omega = -\frac{\mathrm{d}(t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n})}{b_e - t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}}.$$

Observe that

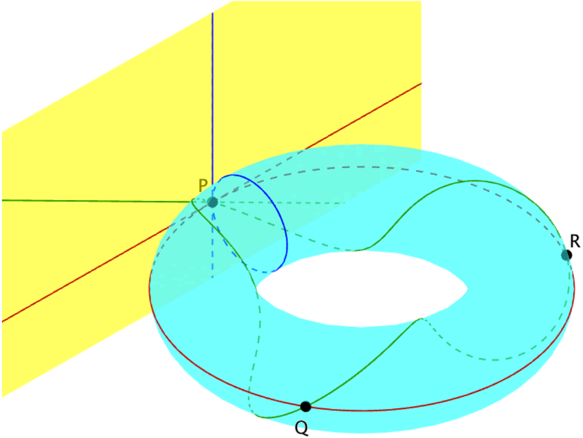
$$\omega_1 \cdot \omega_2 = \omega_{P;1,2} + \omega_{Q;1,2} + \omega_{R;1,2};$$

these two-forms are linearly independent.

Remark

Because intersections of hypertori are, in general, not connected, the cohomology algebra is not always generated in degree one.

We consider only forms $\omega_{W;A}$ where W is a c.c. of $\cap_{a \in A} D_a$.



Consider the exponential map $\mathrm{T}_P T \to T$ and its pullback $H^\bullet(M(\mathcal{A})) \twoheadrightarrow H^\bullet(M(\mathcal{A}[P]))$.
The forms $\omega_{Q;1,2}$, $\omega_{R;1,2}$ and those in $H^\bullet(T)$ belong to the kernel.

The cohomology module

By the results about hyperplane we have:

$$\omega_{P;12} - \omega_{13} + \omega_{23} \equiv 0 \quad (H^1(T)).$$

and more generally for L c.c. of $\cap_{c \in C} D_c$

$$\partial \omega_{L,C} := \sum_{i=0}^{k} (-1)^i \omega_{W_i; C \setminus c_i} \equiv 0 \quad (H^1(T)). \tag{1}$$

The cohomology module

By the results about hyperplane we have:

$$\omega_{P;12} - \omega_{13} + \omega_{23} \equiv 0 \quad (H^1(T)).$$

and more generally for L c.c. of $\cap_{c \in C} D_c$

$$\partial \omega_{L,C} := \sum_{i=0}^{k} (-1)^i \omega_{W_i; C \setminus c_i} \equiv 0 \quad (H^1(T)). \tag{1}$$

Theorem (De Concini, Procesi 2005)

The graded ring $\mathrm{gr}_{(H^1(T))} H^\bullet(M(\mathcal{A}))$ *is generated by* $\omega_{W,A}\psi$, *where* ψ *is any element in* $H^\bullet(W)$ *with the relations of eq.* (1) *and multiplication given by*

$$\omega_{W;A}\omega_{W';A'} = \pm \sum_{L \text{ c.c. } W \cap W'} \omega_{L;AA'}.$$

Analogous results are given by Bibby ('15) and Callegaro, Delucchi ('15) by using spectral sequences.

Unimodular case

A toric arrangement is *unimodular* if all intersections $\cap_{i \in A} D_a$ are empty or connected.
We choose $f_W = 1$, so $\omega_{W;A} = \omega_{a_1} \cdots \omega_{a_q}$ and define $\psi_e = \chi_e^*(\omega) \in H^1(T)$.

Unimodular case

A toric arrangement is *unimodular* if all intersections $\cap_{i\in A} D_a$ are empty or connected.
We choose $f_W = 1$, so $\omega_{W;A} = \omega_{a_1}\cdots\omega_{a_q}$ and define $\psi_e = \chi_e^*(\omega) \in H^1(T)$.

Theorem (De Concini, Procesi 2005)

If $\chi_0 = \chi_1 + \cdots + \chi_q$, then for the circuit $C = (0, 1, \ldots, q)$ the following relation in cohomology holds:

$$\partial\omega_C = \sum_{\substack{0\in A\\ B\neq\emptyset}} (-1)^{\epsilon(A)}\omega_A\psi_B.$$

Unimodular case

A toric arrangement is *unimodular* if all intersections $\cap_{i\in A} D_a$ are empty or connected.
We choose $f_W = 1$, so $\omega_{W;A} = \omega_{a_1} \cdots \omega_{a_q}$ and define $\psi_e = \chi_e^*(\omega) \in H^1(T)$.

Theorem (De Concini, Procesi 2005)

If $\chi_0 = \chi_1 + \cdots + \chi_q$, then for the circuit $C = (0, 1, \dots, q)$ the following relation in cohomology holds:

$$\partial \omega_C = \sum_{\substack{0\in A \\ B\neq\emptyset}} (-1)^{\epsilon(A)} \omega_A \psi_B.$$

Proof.
Follows from the polynomial identity:

$$1 - \prod_{i=1}^{q} x_i = \sum_{I \subsetneq [q]} \prod_{i\in I} x_i \prod_{j\notin I} (1 - x_j).$$ □

When $\chi_0 + \chi_1 + \cdots + \chi_q = 0$ the following factorization holds:

$$\prod_{i=1}^{q} (\omega_i - \omega_{i-1} + \psi_{i-1}) = 0$$

When $\chi_0 + \chi_1 + \cdots + \chi_q = 0$ the following factorization holds:

$$\prod_{i=1}^{q} (\omega_i - \omega_{i-1} + \psi_{i-1}) = 0$$

If we use $-\chi$ to describe D, then we have the transformations:

$$\omega' = \omega - \psi \qquad \psi' = -\psi$$

When $\chi_0 + \chi_1 + \cdots + \chi_q = 0$ the following factorization holds:

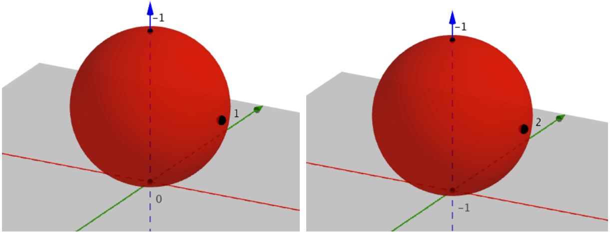
$$\prod_{i=1}^{q} (\omega_i - \omega_{i-1} + \psi_{i-1}) = 0$$

If we use $-\chi$ to describe D, then we have the transformations:

$$\omega' = \omega - \psi \qquad \psi' = -\psi$$

Choose now the following canonical form for every hyperplane D_e:

$$\overline{\omega}_e := \omega_e + \omega'_e = 2\omega_e - \psi_e = \frac{x_e + 1}{x_e(x_e - 1)} \mathrm{d}x_e$$



(a) Residues of ω_e (b) Residues of $\overline{\omega}_e$

Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. 2017)

If $\mathcal{A}$ is unimodular, the relations in cohomology are:

$$\prod_{i=1}^{q} (\overline{\omega}_i + c_i\psi_i - \overline{\omega}_{i-1} + c_{i-1}\psi_{i-1}) = 0,$$

where $\sum_i c_i\chi_i = 0$, $c_i = \pm 1$ *or, equivalently:*

$$\sum_{j=0}^{k} \sum_{A \not\ni j} (-1)^{|A_{\leq j}|} c_B \overline{\omega}_A \psi_B = 0.$$

Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. 2017)

If $\mathcal{A}$ is unimodular, the relations in cohomology are:

$$\prod_{i=1}^{q} (\overline{\omega}_i + c_i\psi_i - \overline{\omega}_{i-1} + c_{i-1}\psi_{i-1}) = 0,$$

where $\sum_i c_i\chi_i = 0$, $c_i = \pm 1$ *or, equivalently:*

$$\sum_{j=0}^{k} \sum_{A \not\ni j} (-1)^{|A_{\leq j}|} c_B \overline{\omega}_A \psi_B = 0.$$

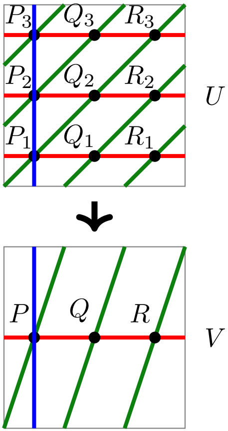
Notice that $c_i\psi_i \in H^1(T)$ does not depend on the choice between χ_i and $-\chi_i$. A central arrangement is invariant for $z \mapsto z^{-1}$, hence:

$$\sum_{j=0}^{q} \sum_{\substack{A \not\ni j \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} c_B \overline{\omega}_A \psi_B = 0. \tag{2}$$

Coverings

Consider the covering $U \to T$ of the tori $u = x$, $v^3 = y$. The hypertori lift to:

$$1 - y \quad \mapsto \quad 1 - v^3$$
$$1 - x^3 y \quad \mapsto \quad 1 - u^3 v^3$$



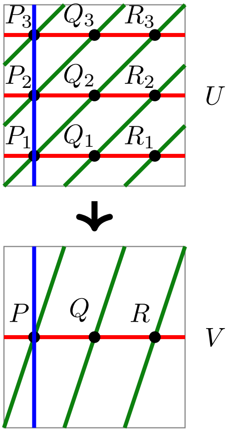
Coverings

Consider the covering $U \to T$ of the tori $u = x$, $v^3 = y$. The hypertori lift to:

$$\begin{array}{ccc} 1-y & \mapsto & 1-v^3 \\ 1-x^3y & \mapsto & 1-u^3v^3 \end{array}$$

A form in U is

$$\overline{\omega}^U_{P_1;1,2} = \overline{\omega}^U_1 \overline{\omega}^U_2 = \frac{v+1}{v(v-1)} \frac{uv+1}{uv(uv-1)} v \mathrm{d}u \mathrm{d}v,$$



Coverings

Consider the covering $U \to T$ of the tori $u = x$, $v^3 = y$. The hypertori lift to:

$$\begin{aligned} 1 - y &\quad\mapsto\quad 1 - v^3 \\ 1 - x^3 y &\quad\mapsto\quad 1 - u^3 v^3 \end{aligned}$$

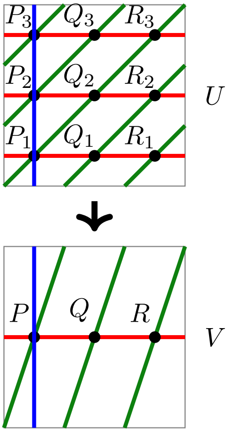
A form in U is

$$\overline{\omega}^U_{P_1;1,2} = \overline{\omega}^U_1 \overline{\omega}^U_2 = \frac{v+1}{v(v-1)} \frac{uv+1}{uv(uv-1)} v \mathrm{d}u \mathrm{d}v,$$

its pushforward is:

$$\overline{\omega}_{P;1,2} := \overline{\omega}^U_{P_1;1,2} + \overline{\omega}^U_{P_2;1,2} + \overline{\omega}^U_{P_3;1,2} =$$

$$= 3 \frac{u^3v^6 + u^3v^3 + 4u^2v^3 + 4uv^3 + v^3 + 1}{uv\left(v^3 - 1\right)\left(u^3v^3 - 1\right)} \mathrm{d}u \mathrm{d}v$$



Coverings

Consider the covering $U \to T$ of the tori $u = x$, $v^3 = y$. The hypertori lift to:

$$\begin{aligned} 1 - y &\mapsto 1 - v^3 \\ 1 - x^3 y &\mapsto 1 - u^3 v^3 \end{aligned}$$

A form in U is

$$\overline{\omega}^U_{P_1;1,2} = \overline{\omega}^U_1 \overline{\omega}^U_2 = \frac{v+1}{v(v-1)} \frac{uv+1}{uv(uv-1)} v \mathrm{d}u \mathrm{d}v,$$

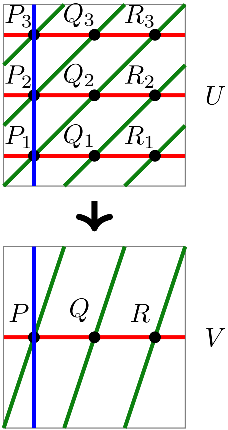
its pushforward is:

$$\overline{\omega}_{P;1,2} := \overline{\omega}^U_{P_1;1,2} + \overline{\omega}^U_{P_2;1,2} + \overline{\omega}^U_{P_3;1,2} =$$

$$= 3 \frac{u^3v^6 + u^3v^3 + 4u^2v^3 + 4uv^3 + v^3 + 1}{uv\,(v^3-1)\,(u^3v^3-1)} \mathrm{d}u \mathrm{d}v$$

$$= \frac{x^3y^2 + x^3y + 4x^2y + 4xy + y + 1}{xy\,(y-1)\,(x^3y-1)} \mathrm{d}x \mathrm{d}y$$

$$= f_P \overline{\omega}_1 \overline{\omega}_2$$



In general:

Lemma

The form $\overline{\omega}_{W,A} = f_W \omega_A$ *does not depend on the covering.*

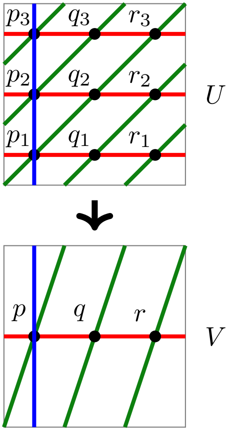
In general:

Lemma

The form $\overline{\omega}_{W,A} = f_W \omega_A$ does not depend on the covering.

Remember eq. (2)

$$\sum_{j=0}^{k} \sum_{\substack{A \not\ni j \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} c_B \overline{\omega}_A \psi_B = 0 \qquad (2)$$



In general:

Lemma

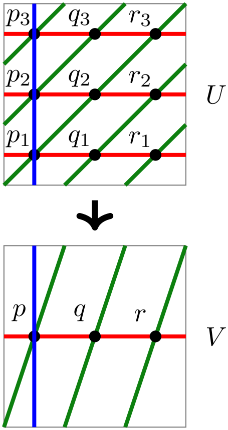
The form $\overline{\omega}_{W,A} = f_W \omega_A$ does not depend on the covering.

Remember eq. (2)

$$\sum_{j=0}^{k} \sum_{\substack{A \not\ni j \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} c_B \overline{\omega}_A \psi_B = 0 \qquad (2)$$

whose pushfoward is

$$\sum_{j=0}^{k} \sum_{\substack{A \not\ni j \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} \frac{m(A)}{m(A \sqcup B)} c_B \overline{\omega}_{W,A} \psi_B = 0.$$



In our example the formula is:

$$\frac{x^3y^2+x^3y+4x^2y+4xy+y+1}{xy\left(y-1\right)\left(x^3y-1\right)}\mathsf{d}x\mathsf{d}y-\frac{x+1}{x(x-1)}\frac{y+1}{y(y-1)}\mathsf{d}x\mathsf{d}y+$$
$$+\frac{x+1}{x(x-1)}\frac{x^3y+1}{y(x^3y-1)}\mathsf{d}x\mathsf{d}y-\frac{1}{3}\,\mathsf{dlog}\,y\;\mathsf{dlog}\,x^3y+\mathsf{dlog}\,y\;\mathsf{dlog}\,x+$$
$$-\,\mathsf{dlog}\,x^3y\;\mathsf{dlog}\,x=0.$$

Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. 2018)

The ring $H^\bullet(M(\mathcal{A});\mathbb{Q})$ is generated by $\overline{\omega}_{W;A}$, ψ_e with relations for every circuit C and c.c. L of C:

$$\sum_{j=0}^{k} \sum_{\substack{A \not\ni j \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} c_B \frac{m(A)}{m(A \sqcup B)} \overline{\omega}_{W;A} \psi_B = 0$$

$$\overline{\omega}_{W,A} \overline{\omega}_{W';A'} = \pm \sum_{L \text{ c.c. } W \cap W'} \overline{\omega}_{L;AA'}.$$

Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. 2018)

The ring $H^\bullet(M(\mathcal{A});\mathbb{Q})$ is generated by $\overline{\omega}_{W;A}$, ψ_e with relations for every circuit C and c.c. L of C:

$$\sum_{j=0}^{k} \sum_{\substack{A \not\ni j \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} c_B \frac{m(A)}{m(A \sqcup B)} \overline{\omega}_{W;A} \psi_B = 0$$

$$\overline{\omega}_{W,A} \overline{\omega}_{W';A'} = \pm \sum_{L \text{ c.c. } W \cap W'} \overline{\omega}_{L;AA'}.$$

Remark

The rational cohomology ring depends only on the poset of layers $\mathcal{S}(\mathcal{A})$ of the toric arrangement.

Example: The two toric arrangements described by

$$N_1 = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix} \text{ and } N_2 = \begin{pmatrix} 1 & 4 & 1 & 6 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix},$$

have different posets of layers and different rational cohomology algebra. However, they describe the same arithmetic matroid and the same matroid over $\mathbb{Z}$.

Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. 2018)

The integral cohomology ring $H^\bullet(M(\mathcal{A});\mathbb{Z})$ is generated by the forms $\omega_{W;A}$ and those in $H^1(T;\mathbb{Z})$.

Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. 2018)

The integral cohomology ring $H^\bullet(M(\mathcal{A}); \mathbb{Z})$ is generated by the forms $\omega_{W;A}$ and those in $H^1(T; \mathbb{Z})$.

Example: The two toric arrangements described by

$$N_3 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 7 & 7 \end{pmatrix} \text{ and } N_4 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 7 \end{pmatrix},$$

have the same poset of layers but different integral cohomology ring.

Thanks for listening!

roberto.pagaria@gmail.com