Roberto Pagaria

Università di Bologna

Cohomology Rings of Toric Wonderful Models

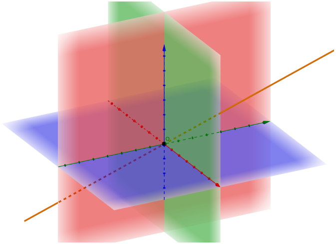
Tropical Days In Bristol

Work in progress with L. Giordani and V. Siconolfi

May 19th, 2023

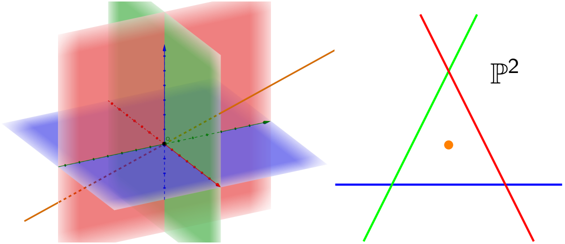
Definition

A *subspace arrangement* in a complex vector space V is a finite collection $\mathcal{A} = \{S_1, \ldots, S_n\}$ of linear subspaces S_i of V.
The *complement* is the open set $M_{\mathcal{A}} = V \setminus \cup\mathcal{A}$.



Definition

A *subspace arrangement* in a complex vector space V is a finite collection $\mathcal{A} = \{S_1, \ldots, S_n\}$ of linear subspaces S_i of V.
The *complement* is the open set $M_{\mathcal{A}} = V \setminus \cup\mathcal{A}$.



Sometimes is useful to work with the projective version: the collection of $\mathbb{P}(S_i) \subset \mathbb{P}(V)$.

Goal

Understand the homotopy type of the complement $M_{\mathcal{A}}$.

The poset of flats

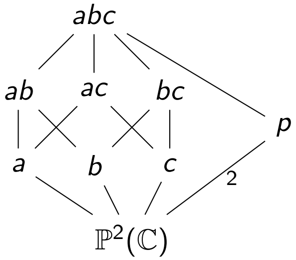
Definition (Poset of flats)

The combinatorial object associated with $\mathcal{A}$ is the *poset of flats*

$$\mathcal{L}_{\mathcal{A}} = \{\cap_{i \in I} S_i \mid I \subseteq [n] := \{1, 2, \ldots, n\}\}$$

of intersections ordered by reverse inclusion, together with the codimension function

$$\mathrm{cd}\colon \mathcal{L}_{\mathcal{A}} \to \mathbb{N}.$$



The poset of flats

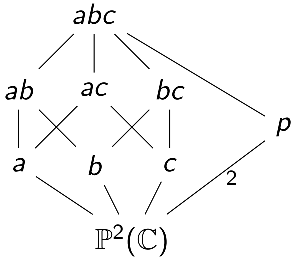
Definition (Poset of flats)

The combinatorial object associated with $\mathcal{A}$ is the *poset of flats*

$$\mathcal{L}_\mathcal{A} = \{\cap_{i \in I} S_i \mid I \subseteq [n] := \{1, 2, \dots, n\}\}$$

of intersections ordered by reverse inclusion, together with the codimension function

$$\mathrm{cd} \colon \mathcal{L}_\mathcal{A} \to \mathbb{N}.$$



Goal

Describe the homotopy type of the complement $M_\mathcal{A}$ in term of the combinatorics $\mathcal{L}_A$, cd.

The poset $\mathcal{L}_\mathcal{A}$ is a lattice.

Example (Rybnicov 1994)

There exist two hyperplane arrangements $\mathcal{A}$ and $\mathcal{A}'$ that have the same combinatorics $\mathcal{L}_{\mathcal{A}} \simeq \mathcal{L}_{\mathcal{A}'}$ but the two complement have different homotopy type $M_{\mathcal{A}} \not\sim M_{\mathcal{A}'}$.

However, the *rational* homotopy type is determine by the combinatorics.

Idea: Construct a nice compactification $Y_{\mathcal{A}}$ of $M_{\mathcal{A}}$ such that $D_{\mathcal{A}} := Y_{\mathcal{A}} \setminus M_{\mathcal{A}}$ is a *simple normal crossing divisor*. The *Morgan model* for the pair $(Y_{\mathcal{A}}, D_{\mathcal{A}})$ is a cdga that codifies the rational homotopy type of the complement $M_{\mathcal{A}}$.

We study the projectified version: regard $\{\mathbb{P}(S_i)\}_{i=,1\dots,n}$ as a projective arrangement in $\mathbb{P}(V)$.
We choose some layers, i.e. a subset $\mathcal{G} \subseteq \mathcal{L}_{\mathcal{A}} \setminus \{\hat{0}\}$ and we totally order $\mathcal{G} = \{G_1, G_2, \dots, G_m\}$. Blow-up the chosen intersection in the chosen order:

$$Y_{\mathcal{A},\mathcal{G}} := \mathrm{Bl}_{G_m}(\mathrm{Bl}_{G_{m-1}}(\dots(\mathrm{Bl}_{G_1}(\mathbb{P}(V))\dots)$$

Remark

The variety $Y_{\mathcal{A},\mathcal{G}}$ is smooth, projective and contains $\mathbb{P}(M_{\mathcal{A}})$.

Definition

A subset $\mathcal{G} \subseteq \mathcal{L}_{\mathcal{A}} \setminus \{\hat{0}\}$ is a *building set* if for any $x \in \mathcal{L}$

$$[\hat{0}, x] = \prod_{y \in \max(\mathcal{G}_{\leq x})} [\hat{0}, y]$$

and

$$\mathrm{cd}(x) = \sum_{y \in \max(\mathcal{G}_{\leq x})} \mathrm{cd}(y).$$

Definition

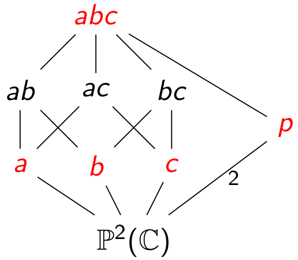
A subset $\mathcal{G} \subseteq \mathcal{L}_{\mathcal{A}} \setminus \{\hat{0}\}$ is a *building set* if for any $x \in \mathcal{L}$

$$[\hat{0}, x] = \prod_{y \in \max(\mathcal{G}_{\leq x})} [\hat{0}, y]$$

and

$$\mathrm{cd}(x) = \sum_{y \in \max(\mathcal{G}_{\leq x})} \mathrm{cd}(y).$$

Example



Definition

A *simple normal crossing divisor* is a divisor whose irreducible components are smooth and intersect locally as coordinate hyperplanes.

Definition

A *simple normal crossing divisor* is a divisor whose irreducible components are smooth and intersect locally as coordinate hyperplanes.

Theorem (De Concini, Procesi 1995)

If $\mathcal{G}$ is a building set, then the divisor $D_{\mathcal{A},\mathcal{G}} := Y_{\mathcal{A},\mathcal{G}} \setminus \mathbb{P}(M_{\mathcal{A}})$ is simple normal crossing with irreducible components $\{K_G\}_{G\in\mathcal{G}}$ the exceptional divisors.

The variety $Y_{\mathcal{A},\mathcal{G}}$ is called *wonderful model* for the subspace arrangement $\mathcal{A}$.

The simple normal crossing divisor $Y_{\mathcal{G}} \setminus M$ has irreducible components $\{K_W\}_{W \in \mathcal{G}}$ in bijections with the building set $\mathcal{G}$.

The simple normal crossing divisor $Y_{\mathcal{G}} \setminus M$ has irreducible components $\{K_W\}_{W \in \mathcal{G}}$ in bijections with the building set $\mathcal{G}$.

Definition (Nested set)

A set $S \subseteq \mathcal{G}$ is $\mathcal{G}$-*nested* if for any non-trivial antichain $T \subseteq S$ we have $\bigvee T \notin \mathcal{G}$. The set $n(\mathcal{G})$ of all $\mathcal{G}$-nested sets is an abstract simplicial complex, called the *nested set complex*.

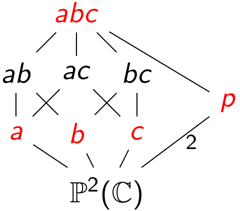
The simple normal crossing divisor $Y_{\mathcal{G}} \setminus M$ has irreducible components $\{K_W\}_{W \in \mathcal{G}}$ in bijections with the building set $\mathcal{G}$.

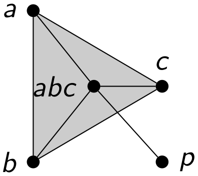
Definition (Nested set)

A set $S \subseteq \mathcal{G}$ is $\mathcal{G}$-*nested* if for any non-trivial antichain $T \subseteq S$ we have $\bigvee T \notin \mathcal{G}$. The set $n(\mathcal{G})$ of all $\mathcal{G}$-nested sets is an abstract simplicial complex, called the *nested set complex*.

The intersection $\cap_{G \in S} K_G$ is non-empty if and only if S is $\mathcal{G}$-nested.

Example





Theorem (De Concini, Procesi 1995)

The cohomology algebra $H^*(Y_{\mathcal{A},\mathcal{G}};\mathbb{Z})$ *is* $\mathbb{Z}[t_G]_{G\in\mathcal{G}}/I$ *where* I *is generated by*

$$\prod_{i=1}^{k} t_{F_i}\Big(\sum_{H\geq G} t_H\Big)^{\mathrm{cd}(G)-\mathrm{cd}(\vee_i F_i)}$$

for any $k\in\mathbb{N}$, $F_1,\ldots,F_k\in\mathcal{G}$, *and* $G\in\mathcal{G}$ *such that* $G\geq \vee_i F_i$.

Consider the *commutative differential graded algebra*

$$\bigoplus_{S\subset\mathcal{G}} H^*\Big(\bigcap_{G\in S} K_G;\mathbb{Q}\Big)$$

with multiplication given by restriction and cup product and the differential given by the Gysin morphism.

Theorem (Morgan 1978)

This algebra is a rational model for $\mathbb{P}(M(\mathcal{A}))=Y_{\mathcal{A},\mathcal{G}}\setminus D_{\mathcal{A},\mathcal{G}}$. *In particular, its cohomology is* $H^*(\mathbb{P}(M(\mathcal{A}));\mathbb{Q})$.

Toric arrangements

Let $T \simeq (\mathbb{C}^*)^r$ be an algebraic torus with character group $\Lambda \simeq \mathbb{Z}^r$. A *layer* G is a connected subtorus, i.e. there exists a *split* direct summand $\Gamma \subseteq \Lambda$ such that

$$G = \{t \in T \mid \chi(t) = 1 \,\forall \chi \in \Gamma\}.$$

Definition (Toric arrangement)

A *toric arrangement* $\mathcal{A} = \{S_1, \dots, S_n\}$ is a finite collection of layers $S_i \subsetneq T$. The *complement* is the open set $M_{\mathcal{A}} = T \setminus \cup\mathcal{A}$.

Definition (Poset of layers)

The combinatorial object associated with $\mathcal{A}$ is the *poset of layers*

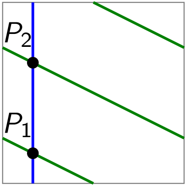
$$\mathcal{L}_{\mathcal{A}} = \bigcup_{I \subseteq [n]} \text{c. c. of } \cap_{i \in I} S_i$$

given by connected components of intersections ordered by reverse inclusion, together with the codimension function

$$\operatorname{cd} \colon \mathcal{L}_{\mathcal{A}} \to \mathbb{N}.$$

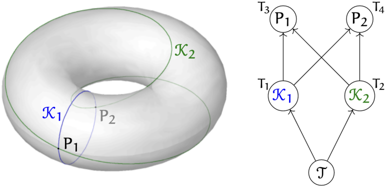
Example

Consider the two hypertori in $T = (\mathbb{C}^*)^2$ defined by:



$$t_1 = 1$$

$$t_1 t_2^2 = 1$$



Papini - PhD thesis

The main difference with subspace arrangements consist in the fact that intersections are not connected. In particular, $\mathcal{L}_{\mathcal{A}}$ is not a semilattice. Moreover, there is no projective version of the algebraic torus.

As a first step we compactify the algebraic torus T.
Let Σ in the cocharacter group of T be a smooth projective fan and X_Σ be the associated toric variety containing T as open orbit.

Definition (Good toric variety)

The toric variety X_Σ is *good* with respect to $\mathcal{A}$ if for every maximal cone C and any layer G_Γ we have $\Gamma \subseteq C^* \cup -C^*$.

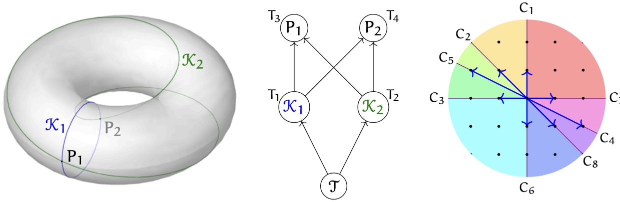
We see the complement $M_\mathcal{A}$ as $X_\Sigma \setminus (\bigcup \mathcal{A} \cup B)$ where $B = X_\Sigma \setminus T$ is the boundary.

Theorem (De Concini Gaiffi 2017)

If X_Σ is a good toric variety then for every layer $G \in \mathcal{A}$ the closure $\overline{G}$ in X_Σ is smooth and $\overline{G}$ intersects transversally every orbit closure $\overline{\mathcal{O}} \subset X_\Sigma$.

Example

A fan Σ whose toric variety X_Σ is good for the arrangement $t_1 = 1$, $t_1 t_2^2 = 1$.



Papini - PhD thesis

Now, we need to blowup the intersections inside the torus T.
We choose some layers, i.e. a subset $\mathcal{G} \subseteq \mathcal{L}_{\mathcal{A}} \setminus \{\hat{0}\}$ and we totally order $\mathcal{G} = \{G_1, G_2, \dots, G_m\}$. Blow-up the chosen intersection in the chosen order:

$$Y_{\mathcal{A},\mathcal{G},\Sigma} := \mathrm{Bl}_{\overline{G}_m}(\mathrm{Bl}_{\overline{G}_{m-1}}(\dots(\mathrm{Bl}_{\overline{G}_1}(X_\Sigma)\dots))$$

Definition

A subset $\mathcal{G} \subseteq \mathcal{L}_{\mathcal{A}} \setminus \{\hat{0}\}$ is a *building set* if for any $x \in \mathcal{L}$ the set $\mathcal{G}_{\leq x} := \{G \in \mathcal{G} \mid G \leq x\}$ is a building set in the lattice $\mathcal{L}_{\leq x}$.

Theorem (De Concini, Gaiffi 2017)

If $\mathcal{G}$ is a building set, then the divisor $D_{\mathcal{A},\mathcal{G}} := Y_{\mathcal{A},\mathcal{G},\Sigma} \setminus M_{\mathcal{A}}$ is simple normal crossing with irreducible components $\{K_G\}_{G\in\mathcal{G}}$ the exceptional divisors and the strict transform of orbit closure $\tilde{\mathcal{O}}_r$ for each ray $r \in \Sigma$.

The variety $Y_{\mathcal{A},\mathcal{G},\Sigma}$ is called *toric wonderful model* for the arrangement $\mathcal{A}$.

For $S \subseteq \mathcal{L}_{\mathcal{A}}$ define the join $\vee S$ as the set of all least upper bound of S in $\mathcal{L}_{\mathcal{A}}$. In particular $\vee S$ is the set of connected components of $\cap_{G \in S} G$.

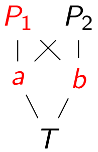
Definition

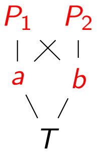
A building set $\mathcal{G}$ is *well connected* if for any $S \subseteq \mathcal{G}$ either $|\vee S| \leq 1$ or $\vee S \subseteq \mathcal{G}$.

For well connected building set $\mathcal{G}$ they define the $\mathcal{G}$-nested set as $S \subset \mathcal{G}$ such that for any (or for all) $x \in \vee S$ the set S is $\mathcal{G}_{\leq x}$-nested in $\mathcal{L}_{\leq x}$.

Example

The previous toric arrangement with two possible building sets $\mathcal{G}$:





Let B_Σ be the ring $\mathbb{Q}[c_r \mid r \in \Sigma]$ with relations

- $c_{r_1} c_{r_2} \dots c_{r_l} = 0$ if $r_1, r_2, \dots, r_l$ do not form a cone of Σ (Stanley-Reisner relations).
- $\sum_{r \in \Sigma} m(u_r) c_r = 0$ for any character m and u_r the primitive vector in the ray r.

Theorem (Jurkiewicz-Danilov 1978)

Let X_Σ be smooth and projective. Then $H^*(X_\Sigma; \mathbb{Q}) \simeq B_\Sigma$.

Theorem (De Concini Gaiffi 2019)

Let $\mathcal{G}$ be well connected, then $H^*(Y_{\mathcal{A},\mathcal{G},\Sigma}; \mathbb{Q}) = B_\Sigma[t_G]_{G \in \mathcal{G}}/I$ where I is generated by

- $t_G c_r$ *if* $r \notin \mathrm{Ann}(\Lambda_G)$ *(equiv.* $\tilde{\mathcal{O}}_r \cap K_G = \emptyset$*),*
- $\prod_{i=1}^k t_{F_i} P_G^{\vee_i F_i}(\sum_{H \geq G} t_H)$ *for any* $k \in \mathbb{N}$*,* $F_1, \dots, F_k \in \mathcal{G}$*, and* $G \in \mathcal{G}$ *such that* $G \geq \vee_i F_i$*.*

Our approach is more general.

Definition (Blowup of posets)

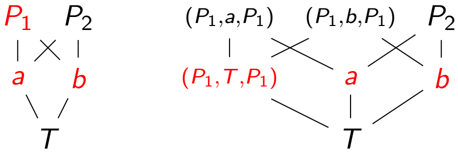
$$\mathrm{Bl}_p\,\mathcal{L} = \{x \mid x \ngeq p\} \sqcup \{(p,x,y) \mid x \ngeq p,\ y \in p \vee x\}$$

with the ordering given by

1. $x \geq x'$ if $x \geq x'$,
2. $(p,x,y) \geq (p,x',y')$ if $x \geq x'$ and $y \geq y'$,
3. $(p,x,y) \geq x'$ if $x \geq x'$.

Example

A poset $\mathcal{L}$ and its blowup $\mathrm{Bl}_{P_1}(\mathcal{L})$ at P_1:



Our approach is more general.

Definition (Blowup of posets)

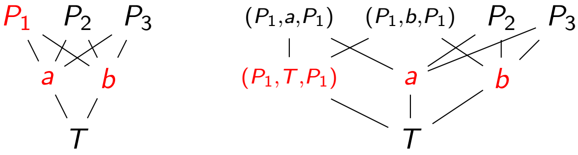
$$\mathrm{Bl}_p \mathcal{L} = \{x \mid x \not\geq p\} \sqcup \{(p, x, y) \mid x \not\geq p,\ y \in p \vee x\}$$

with the ordering given by

1. $x \geq x'$ if $x \geq x'$,
2. $(p, x, y) \geq (p, x', y')$ if $x \geq x'$ and $y \geq y'$,
3. $(p, x, y) \geq x'$ if $x \geq x'$.

Example

A poset $\mathcal{L}$ and its blowup $\mathrm{Bl}_{P_1}(\mathcal{L})$ at P_1:



Let us consider a building set $\mathcal{G} = \{G_1, \dots, G_m\}$ in $\mathcal{L}$, define :

$$\mathrm{Bl}_{\mathcal{G}}(\mathcal{L}) := \mathrm{Bl}_{G_m}(\mathrm{Bl}_{G_{m-1}}(\dots(\mathrm{Bl}_{G_1}(\mathcal{L})\dots))$$

Definition

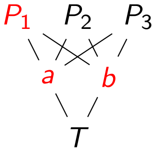
A pair (S, x) is $\mathcal{G}$-*nested* if $x \in \vee S$ and S is $\mathcal{G}_{\leq x}$-nested in $\mathcal{L}_{\leq x}$.
The *nested set complex* $n(\mathcal{G})$ is the Δ-complex of all $\mathcal{G}$-nested sets.

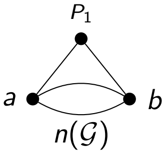
Lemma (Giordani, P., Siconolfi 2023)

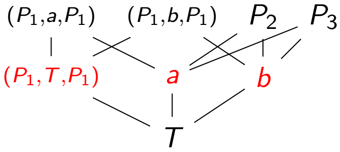
The poset $\mathrm{Bl}_{\mathcal{G}}(\mathcal{L})$ *is the face poset of the* Δ*-complex* $n(\mathcal{G})$*. In particular, it is locally boolean poset.*

Example

A poset $\mathcal{L}$ and the Δ-complex $n(\mathcal{G})$:







There exists a projection map $\pi\colon \mathrm{Bl}_{\mathcal{G}}(\mathcal{L}) \to \mathcal{L}$ induced by $(x, p, y) \mapsto y$.

Theorem (Giordani, P., Siconolfi 2023)

Let $\mathcal{G}$ be a building set, then $H^*(Y_{\mathcal{A},\mathcal{G},\Sigma};\mathbb{Q}) = B_\Sigma[t_a]_{a\in \mathrm{Bl}_{\mathcal{G}}(\mathcal{L})}/I$ with $\deg(t_a) = 2|a| = 2\,\mathrm{rk}(a)$ and I is generated by

- $t_{\pi(a)}c_r$ *if* $r \notin \mathrm{Ann}(\Lambda_{\pi(a)})$ *(equiv.* $\tilde{\mathcal{O}}_r \cap K_a = \emptyset$*),*
- $t_a P_G^{\pi(a)}(\sum_{H\geq G} t_{\{H\}})$ *for any* $a \in \mathrm{Bl}_{\mathcal{G}}(\mathcal{L})$*, and* $G \geq \pi(a)$*.*
- $t_a t_b = t_{a\wedge b} \sum_{c\in a\vee b} t_c$ *for any* $a, b \in \mathrm{Bl}_{\mathcal{G}}(\mathcal{L})$*.*

Thanks for listening!

roberto.pagaria@unibo.it