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Decomposition theorem for Hitchin fibrations and poset of non-integral flats

Geometry, Algebra and Combinatorics of Moduli Spaces and Configurations V

Work in progress with M. Mauri and L. Migliorini

February 21, 2023

- **Algebraic geometry**
  1. The Hitchin fibration
  2. Spectral curve
  3. The Decomposition theorem
- **Combinatorics**
  4. Dual graph and graphic matroids
  5. Shellability
  6. Integral points in zonotopes
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The moduli space $M(n, d)$

Let C be a smooth projective algebraic curve over $\mathbb{C}$ of genus $g_C > 1$. We consider a vector bundle $\mathcal{E}$ of rank n and degree d on C.

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Definition

The *Dolbeault moduli space* is

$$M(n, d) = {}^{\{\text{semistable Higgs bundle}\}}\big/_{S\text{-equivalence}}.$$

Hitchin fibration

Every endomorphisms has a characteristic polynomial.
For an Higgs bundle $(\mathcal{E}, \phi)$ we consider the characteristic polynomial

$$\begin{aligned}\chi_\phi(t) &= t^n + a_1 t^{n-1} + \cdots + a_n \\ &= t^n - \mathrm{tr}(\phi) t^{n-1} + \cdots + (-1)^n \det(\phi)\end{aligned}$$

where $a_i \in H^0(C, \omega_C^{\otimes i})$. Define $A_n = \bigoplus_{i=1}^n H^0(C, \omega_C^{\otimes i}) \simeq \mathbb{A}^N$.

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Definition

The *Hitchin fibration* is the map

$$\chi \colon M(n, d) \to A_n$$

sending $(\mathcal{E}, \phi)$ to $(a_1, a_2, \ldots, a_n)$.

The base of the fibration does not depend on d!

Spectral curve

For any point $a \in A_n$ the associated characteristic polynomial $p_a(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ describes a curve in the cotangent bundle $T^*C \to C$.

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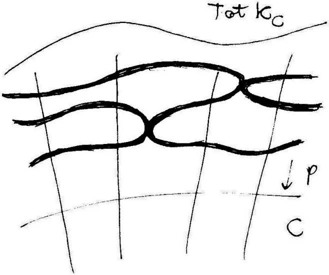
Definition

This zero locus is called *spectral curve* C_a.



$C_a = (\mathrm{char}\,\phi = 0)$, F = "sheaf of eigenvectors"

$(E, \phi) = (p_* F, y\cdot)$

Beauville Narasimhan Ramanan correspondence

Lemma

The BNR correspondence is

$$\chi^{-1}(a) = \{(\mathcal{E}, \phi) \mid \chi_\phi = p_a\} \longleftrightarrow \begin{Bmatrix} \textit{semistable rank one torsion} \\ \textit{free sheaves on } C_a \end{Bmatrix}$$

obtained by pushforward along $C_a \to C$.

The dimension of $M(n, d)$ is $2(g_C - 1)n^2 + 2$.

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We study the behaviour only on the *reduced locus* $A_{n,\mathrm{red}} \subset A_n$ where the corresponding polynomial $p_a(t)$ has distinct irreducible factors. For any partition $\underline{n} = (n_1, n_2, \ldots, n_r) \vdash n$ we define $S_{\underline{n}} \subset A_{n,\mathrm{red}}$ the set of points a such that the irreducible factors of $p_a(t)$ have degree $(n_1, n_2, \ldots, n_r)$.

The strata $S_{\underline{n}}$ form a Whitney stratification of $A_{n,\mathrm{red}}$.

Decomposition theorem

The space $M(n,d)$ is singular with a map to the affine space A_n.
The cohomology does not work well on singular spaces, it is much better to consider the *intersection cohomology* $\mathrm{IH}(M(n,d))$.

$$\mathrm{IH}(M(n,d)) = H(M(n,d), \mathrm{IC}_{M(n,d)}) \simeq H(A_n, R\chi_* \mathrm{IC}_{M(n,d)})$$

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where IC is the *perverse intersection complex*.

Theorem (Mauri, Migliorini '22)

The Decomposition Theorem specializes to

$$R\chi_* \mathrm{IC}_{M(n,d)}|_{A_{\mathrm{red}}} = \bigoplus_{\underline{n} \vdash n} \mathrm{IC}_{S_{\underline{n}}}(\mathcal{L}_{\underline{n},d} \otimes \Lambda_{\underline{n}})$$

for some local systems $\mathcal{L}_{\underline{n},d}$ on $S_{\underline{n}}$ and for $\Lambda_{\underline{n}}$ the cohomology sheaf of the relative Picard group $\mathrm{Pic}^0(\overline{C}_{\underline{n}})$ of the normalization of the spectral curve.

Main problem

Problem: determine $\mathcal{L}_{\underline{n},d}$. In particular:

1. which partitions $\underline{n}$ appear in the decomposition (i.e. $\mathcal{L}_{\underline{n},d} \neq 0$)?
2. determine the rank $r(\underline{n}, d) := \dim(\mathcal{L}_{\underline{n},d})_a$.
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Remark

For the trivial partition $(n) \vdash n$

$$\mathcal{L}_{(n),d} = \mathbb{Q}$$

The dual graph

For $a \in A_{n,\mathrm{red}}$ the spectral curve is reduced. If $a \in S_{\underline{n}}$ the spectral curve has $r = \ell(\underline{n})$ irreducible components of degree n_i. The number of intersection points of two irreducible components is

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Let $\Gamma_{\underline{n}} = \Gamma_a$ be the *dual graph* of the spectral curve C_a, i.e. the graph on r vertices and $y_{i,j} := n_i n_j (2g_C - 2)$ edges between the vertices i and j.
Define the vector $\omega = (\frac{dn_1}{n}, \frac{dn_2}{n}, \ldots, \frac{dn_r}{n}) \in \mathbb{Q}^r$.

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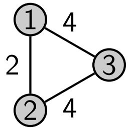
Example

Let $n = 4$, $d = 2$, $g = 2$, and $\underline{n} = (1, 1, 2)$.
The dual graph is :



$$\omega = \left(\frac{1}{2}, \frac{1}{2}, 1\right)$$

Definition

The *graphical zonotope* Z_Γ of Γ is the integral polytope defined by

$$Z_\Gamma := \sum_{(i,j)\in\Gamma} y_{i,j}[0, e_i - e_j] \subset \mathbb{R}^{V(\Gamma)}$$

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For any polytope Z let $C(Z)$ be the number of integral points in the interior of Z.

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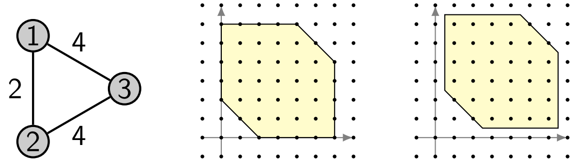

so $C(Z_\Gamma) = 23$ and $C(Z_\Gamma + \omega) = 30$.

From Algebraic Geometry to Combinatorics

Proposition

For any $a \in S_{\underline{n}}$ we have

$$\dim \mathcal{H}^{\text{top}}(R\chi_* \mathrm{IC}_{M(n,d)})_a = C(Z_{\Gamma_{\underline{n}}} + \omega)$$

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Recall the BNR correspondence and that the multidegree function has irreducible fiber:

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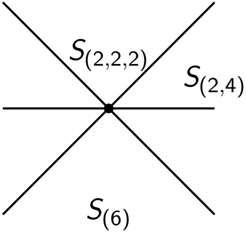
Example

The strata $S_{(2,2,2)}$ is contained in the strata $S_{(2,4)}$, however there are three branching of $S_{(2,4)}$ concurring at $S_{(2,2,2)}$. Let $a \in S_{(2,2,2)}$, then $p_a(t) = p_1(t)p_2(t)p_3(t)$ is the product of three distinct polynomials of degree two. So the branching are in correspondence with the *set partitions* of [3], i.e. the ways to multiply some of its factors.



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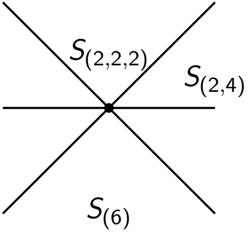

Definition

Let $\underline{n} \vdash n$ and $\underline{S} \vdash [\ell(\underline{n})]$. Define the partition $\underline{n}_{\underline{S}} \vdash n$ as $(\sum_{j \in S_i} n_j)_i$.

Theorem (Mauri, Migliorini '22)

The Decomposition Theorem specializes to

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For $a \in S_{\underline{n}}$ we have

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and so

$$C(Z_{\Gamma_{\underline{n}}} + \omega) = \sum_{\underline{S} \vdash [\ell(\underline{n})]} r(\underline{n}_{\underline{S}}, d) \prod_{i=1}^{\ell(\underline{S})} C(Z_{\Gamma_{S_i}})$$

where Γ_S is the induced subgraph of Γ on the vertices $S \subseteq V$.

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where Γ_S is the induced subgraph of Γ on the vertices $S \subseteq V$. Notice that each face of Z_Γ is of the type $Z_{\Gamma_{S_1}} \times Z_{\Gamma_{S_2}} \times \cdots \times Z_{\Gamma_{S_l}}$ for some set partition of the vertices of Γ.

Graphic matroids

We consider graphs $\Gamma = (V, E)$ possibly with multiple edges and the associated *cycle matroid*.

Cycle matroid	Graph
Groundset	Set of edges
Independent	Forest
Closure	Induced subgraph
Flat	Partition of V with connected blocks

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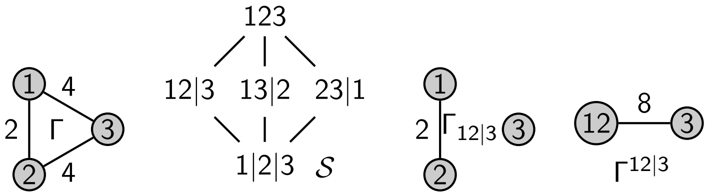
Define the *poset of flats* $\mathcal{S}$.

Definition

Let $\underline{S} \in \mathcal{S}$ be a flat, the *deleted* graph $\Gamma_{\underline{S}}$ is the graph with only edges in the flat $\underline{S}$. The *contracted* graph $\Gamma^{\underline{S}}$ is obtained from Γ by contracting all the edges in the flat $\underline{S}$.

Example

Consider the graph Γ with poset of flats $\mathcal{S}$ and the flat $12|3$.



Counting integral points

Theorem (Stanley '91, Ardila Beck McWhirter '20)

Let $Z = \sum_{i \in E} [0, v_i]$ be an integral zonotope and $\omega \in \mathbb{R}^r$. Then

$$C(Z + \omega) = \sum_{I \text{ independent set}} (-1)^{r-|I|} \delta_{(\langle v_i \rangle_{i \in I} + \omega) \cap \mathbb{Z}^r \neq \emptyset} \operatorname{Vol}(I).$$

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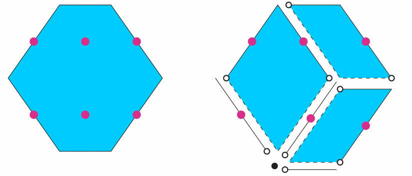
Example

Let $Z = [0, e_1] + [0, e_1 + e_2] + [0, e_1 - e_2]$ and $\omega = (\frac{1}{2}, \frac{1}{2})$.



$$C(Z+\omega) = \operatorname{Vol}(v_2 v_3) + \operatorname{Vol}(v_1 v_2) + \operatorname{Vol}(v_1 v_3) - \operatorname{Vol}(v_2) - \operatorname{Vol}(v_3)$$
$$= 2 + 1 + 1 - 1 - 1 = 2.$$

Ardila, Supina,Vindas-Meléndez - The equivariant Ehrhart theory of the permutahedron

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$$C(Z+\omega) = \sum_{W \text{ flat}} (-1)^{r - \dim W} \delta_{(W+\omega) \cap \mathbb{Z}^r \neq \emptyset} \sum_{\substack{I \text{ independent set} \\ \langle I \rangle = W}} \operatorname{Vol}(I).$$

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Definition

A set $S \subseteq [r]$ is ω-integral if $\sum_{i \in S} \omega_i \in \mathbb{Z}$. A partition $\underline{S} \vdash [r]$ is ω-integral if all its blocks S_j are ω-integral.

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For a graphical zonotope Z_Γ and a flat $\underline{S} \in \mathcal{S}$ we have $\delta_{(\langle \underline{S} \rangle + \omega) \cap \mathbb{Z}^r \neq \emptyset} = 1$ if and only if $\underline{S}$ is ω-integral.

Möbius inversion

Theorem (Mauri, Migliorini, P. '23)

If $\sum_{i=1}^{r} \omega_i \in \mathbb{Z}$*, then*

$$C(Z_\Gamma + \omega) = C(Z_\Gamma) + \sum_{\underline{S} \in \mathcal{S}} \Big(\sum_{\substack{\underline{T} \geq \underline{S} \\ \underline{T}\ \omega\text{-integral}}} \mu(\underline{S}, \underline{T}) \Big) C(Z_{\Gamma_{\underline{S}}}).$$

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Moreover, $\mathcal{L}_{\underline{n},d} = 0$ *if* $\omega \in \mathbb{Z}^r$, *i.e.* $\frac{dn_i}{n} \in \mathbb{Z}$ *for all* i.

Shellability

We denote by $\mathcal{S}_\omega \subset \mathcal{S}$ the downward closed subposet of non-ω-integral flats. Let $\Delta(\mathcal{S}_\omega)$ be the the *order complex* of the poset $\mathcal{S}_\omega$.

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If $\omega \notin \mathbb{Z}^r$, i.e. exists i such that $\frac{dn_i}{n} \notin \mathbb{Z}$, then $\mathcal{L}_{\underline{n},d} \neq 0$.

This solves Problem 1.

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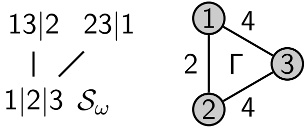
Example

Let $n = 4$, $g = 2$, and $\underline{n} = (1, 1, 2)$. Then $\omega = (\frac{1}{2}, \frac{1}{2}, 1)$ and



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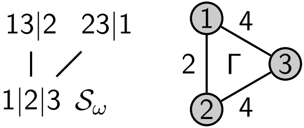

$$C(Z_\Gamma + \omega) = C(Z_\Gamma) + C(Z_{\Gamma_{13|2}}) + C(Z_{\Gamma_{23|1}}) + C(Z_{\Gamma_{1|2|3}})$$
$$30 = 23 + 3 + 3 + 1.$$

Orientation character

Let $O\Gamma$ be the oriented graph obtained by replacing every unoriented edge in Γ with the two possible oriented edges.

Definition

Consider the representation a_Γ of $\mathrm{Aut}(\Gamma)$ defined by

$$a_\Gamma(\sigma) = \mathrm{sgn}(\sigma\colon V(\Gamma) \to V(\Gamma))\,\mathrm{sgn}(\sigma\colon E(O\Gamma) \to E(O\Gamma))$$

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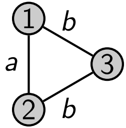

with $a \neq b$. Then $\mathrm{Aut}(\Gamma) = \mathbb{Z}/2\mathbb{Z} = \langle (12) \rangle$ and $a_\Gamma(\sigma) = (-1)^{a+1}$.

Permutation representations

Consider the group $\mathrm{Aut}(\Gamma) < \mathfrak{S}_r$ and suppose that ω is a $\mathrm{Aut}(\Gamma)$-invariant vector. Let $\mathcal{C}(Z_\Gamma + \omega)$ be the permutation representation of $\mathrm{Aut}(\Gamma)$ on the set of integral points in the interior of $Z_\Gamma + \omega$ ($\dim \mathcal{C}(Z_\Gamma + \omega) = C(Z_\Gamma + \omega)$).

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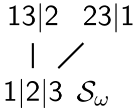

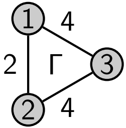

The automorphism group is $\operatorname{Aut}(\Gamma) = \mathbb{Z}/2\mathbb{Z} = \langle (1,2) \rangle$. Then:

$$\mathcal{C}(Z_\Gamma + \omega) = \mathcal{C}(Z_\Gamma) \oplus \mathrm{Reg}^{\oplus 3} \oplus (\mathrm{sgn} \otimes \mathrm{sgn} \otimes 1).$$

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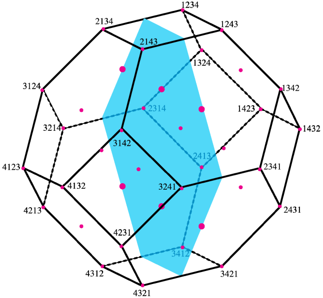

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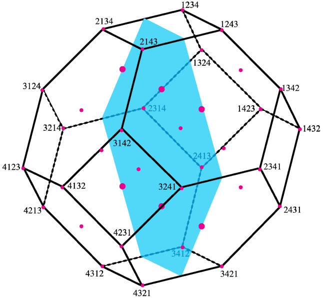

The result follows from

$$C((Z_\Gamma + \omega)^\sigma) = C(Z_\Gamma^\sigma) + \sum_{\underline{S} \in \mathcal{S}^\sigma_\omega} \pm \mu_{\mathcal{S}^\sigma_{\omega, \geq \underline{S}}}(\hat{0}, \hat{1}) C(Z^\sigma_{\Gamma_{\underline{S}}}) \quad \square$$

Ardila, Supina, Vindas-Meléndez - The equivariant Ehrhart theory of the permutahedron

Conclusions

Problem: determine $\mathcal{L}_{\underline{n},d}$. In particular:

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3. The monodromy is given by the representation of $\mathrm{Aut}(\Gamma_{\underline{n}})$ $$\mathrm{sgn} \otimes \widetilde{H}^{\text{top}}(\Delta(\mathcal{S}_\omega)).$$

Thanks for listening!

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