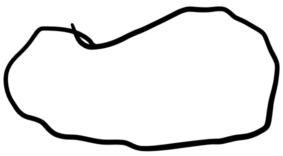
Hitchin fibrations and related combinatorics

PLAN

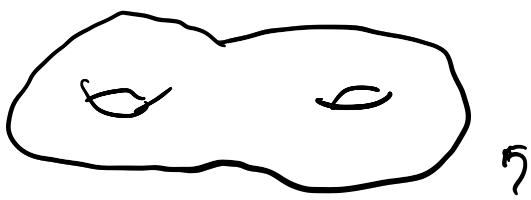
- Moduli space $\qquad \mathcal{M}_{n,d}$
- Hitchin fibration $\qquad \chi \downarrow$
- $\chi^{-1}(a)$ BNR correspondence $\qquad \mathcal{A}_n$
- Intersection Cohomology $\mathcal{M}_{n,d}$

<u>SETTING</u> Let C be a smooth proj alg curve/$\mathbb{C}$ (a Riemann surface)

$\mathbb{C}$



$\mathbb{R}$

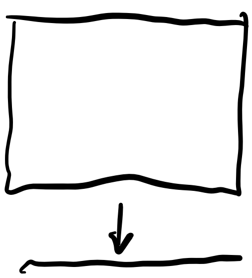


genus of C $\quad g \geq 2$ $\quad$ "number of holes"

<u>DEF</u> a vector bundle E over C of rank n

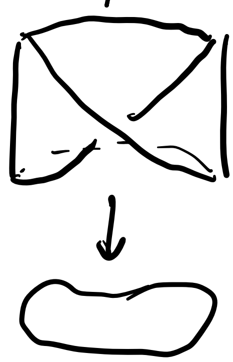
$\mathbb{C}^n$

p



$U \times \mathbb{C}^n$

$U \subset C$



$E \downarrow C$

pointwise $\qquad$ local $\qquad$ global

Operations on vector bundles:

$\oplus$ direct sum, $\otimes$ Tensor product, dual, Λ^n exterior power

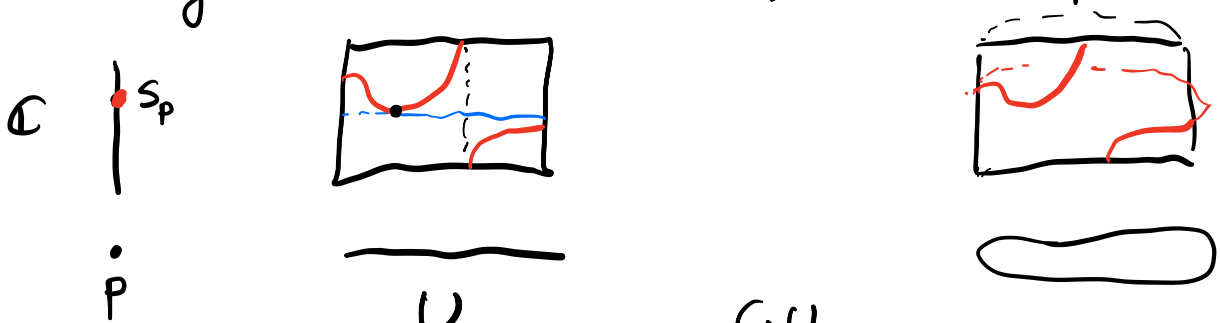
S^λ Schur functors

Eg $\Lambda^n E$ is a line bundle $\boxed{m=1}$
$\quad \overset{\shortparallel}{\det E}$

DEF Let L be a line bundle s a meromorphic section

$$\begin{tikzcd} L \arrow[d] \\ C \arrow[u, bend right, dashed, "s"'] \end{tikzcd}$$

$$\deg L := \#\,\text{zeros}\,(s) - \#\,\text{poles}\,(s)$$



$p \in C \setminus U$

$\deg = 2 - 1 + \dots$

EX $\deg(L)$ does not depend on the choice of s

DEF $\deg(E) := \deg(\det E)$

$\left(\deg E = \int_C c_1(E) \quad \text{first Chern class} \right)$

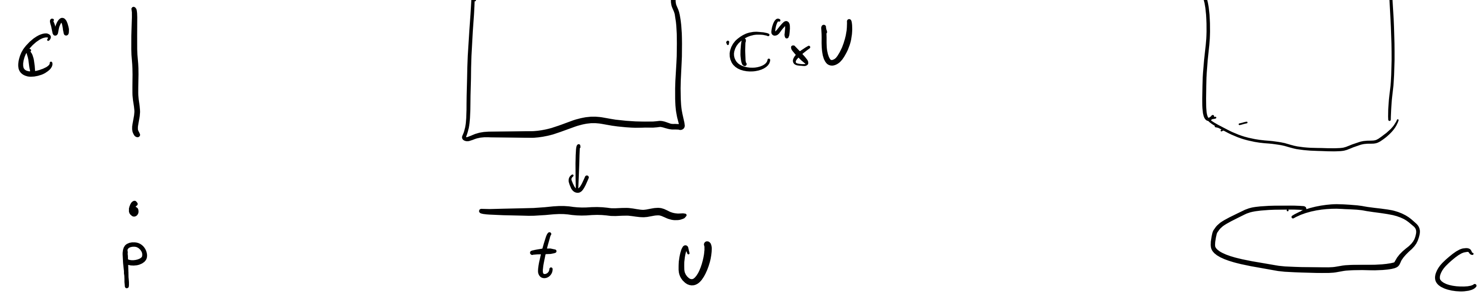
DEF Tangent bundle to X $(= C)$

T^*X be the cotangent bundle

$\overset{\shortparallel}{(TX)^\vee}$ $\qquad \omega_X := \det(T^*X)$ is called <u>canonical</u>

DEF A <u>Higgs bundle</u> is a pair (E, ϕ) where

$\phi : E \to E \otimes \omega_C$



$\phi_p \in Mat_{n,n}(\mathbb{C})$

$\phi|_U$ has entries that depends on t

$U \simeq \mathbb{C} = \mathbb{A}^1$

$\phi|_U \in Mat_{n\times n}(\mathbb{C}[t])$

we glue together the local picture. The transition functions behaves "like" ω_C

GOAL DEFINE the moduli space of Higgs bundles

EX 1 $GL_n \circlearrowright Mat_{n\times n}$ by conjugation

$$Mat_{n,n}/GL_n = \{\text{Jordan normal forms}\}$$

endow with quotient topology

$$\begin{pmatrix} \lambda & 1 & & & & \\ & \ddots & \ddots & & & \\ & & & 1 & t & \\ & & & & 1 & \\ & & & & \ddots & 1 \\ & & & & & \lambda \end{pmatrix} \simeq J_{\lambda,n} \quad \text{for } t \neq 0$$

$n_1 + n_2 = n$

$t \to 0 \quad J_{\lambda,n_1} \oplus J_{\lambda,n_2}$

$$\overline{GL_n \, J_{n,\lambda}} \ni J_{\lambda,n_1} \oplus J_{\lambda,n_2}$$

$\Rightarrow Mat_{n,n}/GL_n$ is not separated (not T_2)

$A \in Mat_{n,n}$ define a closed point $[A]$ iff A is diagonalizable

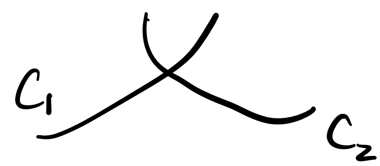
$$J_{\lambda,n} \equiv J_{\lambda,n_1} \oplus J_{\lambda,n_2}$$

$$\begin{array}{ccccccccc} & & \mathbb{C}^{n_1} & & \mathbb{C}^{n} & & \mathbb{C}^{n_2} & & \\ & & \cong & & \cong & & \cong & & \\ 0 & \to & W & \to & V & \to & V/W & \to & 0 \\ & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \\ & & J_{\lambda,n_1} & & J_{\lambda,n} & & J_{\lambda,n_2} & & \end{array}$$

impose $V \equiv W \oplus V/W$

$Mat_{n,n} /\!/ Gl_n$ is an alg var.

Example 2 $\quad C = C_1 \cup C_2$

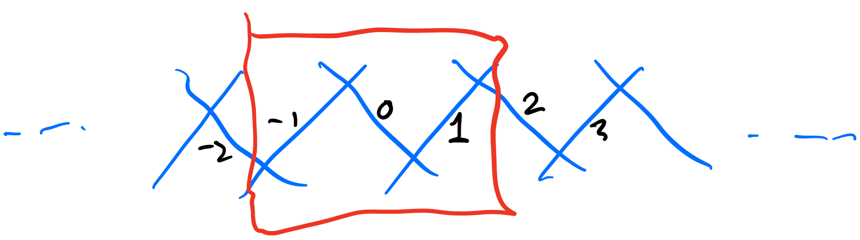


$$\left\{ \begin{array}{c} \text{rank 1 Torsion free sheaves} \\ \text{of degree } d \end{array} \right\} \Big/ \sim$$

$$\downarrow\downarrow \qquad\qquad F \mapsto \deg(F|_{C_1})$$

$$\mathbb{Z}$$

$\deg(F|_{C_2}) = d - \deg(F|_{C_1})$



infinitely many irreducible components

is not of finite Type

Solution Impose a stability condition

Eg $\quad \deg(F|_{C_1}) \le d$

$$\deg(F|_{C_2}) \leq d.$$

DEF (E, ϕ) Higgs bundle

$(F, \phi|_F)$ is a sub Higgs-bundle if

$F \subseteq E$ is a sub bundle

and F is ϕ invariant

$$\mu(E) := \frac{\deg(E)}{\operatorname{rank}(E)} = \frac{d}{n} \in \mathbb{Q} \qquad \text{slope of } E$$

DEF $(E, \cancel{\phi})$ is (semi) stable ~~Higgs~~ bundle if

$\forall\ F \subset E$ sub ~~Higgs~~-bundle

$$\mu(F) \underset{(\leq)}{<} \mu(E)$$

DEF $M^{c}_{n,d} := \{$ semistable Higgs bundle of rank n and degree $d \}$ / S-equivalence

$$0 \to F \to E \to G \to 0 \qquad \text{s.e.s. of semistable Higgs bundles}$$

$$\mu(F) = \mu(E) = \mu(G)$$

$\Rightarrow F \oplus G \equiv E$ S-equivalent

EX $\mu(F) = \mu(E)$ $F \subset E$ sub-bundle, E s.s $\Rightarrow$

F is s.s. $\mu(E/F) = \mu(E)$ E/F is s.s.

REMARK. $\exists\ (E, \phi)$ semistable s.t. E is not stable as vector bundle

- rank, degree are additive on s.e.s.
- $N^{C}_{d,n}$ moduli space of vector bundles

$\ni$

E a s.s. vector bundle

$$T^*_E N_{d,n} \simeq H^1(C, \mathrm{End}(E))^\vee \underset{\text{Serre duality}}{\simeq} H^0(C, \mathrm{End}(E)\otimes \omega_C)$$

$$\ni \quad \phi : E \to E \otimes \omega_C$$

$T^* N_{d,n} \subseteq M_{d,n}$ open set

$(E,\phi) \qquad (E,\phi)$

E is s.s. vector bundle

$\Rightarrow \forall \phi \quad (E,\phi)$ is s.s. Higgs bundle

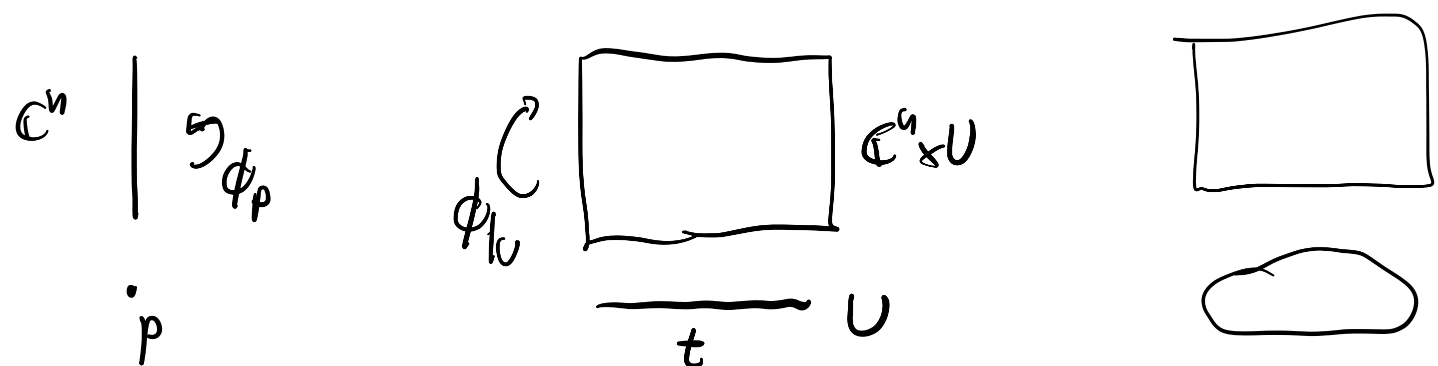
$$\dim M_{d,n} = 2 \dim N_{d,n} = 2n^2(g-1)+2$$

- $M_{d,n}$ is non-compact, singular
- if $(d,n)=1$ then $M_{d,n}$ is smooth

Ex $(d,n)=1$ (E,ϕ) s.s. $\overset{\text{Higgs}}{\Rightarrow}$ (E,ϕ) is stable $^{\text{Higgs}}$

Hitchin fibration

$(E,\phi) \rightsquigarrow \chi_\phi$ characteristic polynomial



$\chi_{\phi_p}(\lambda) \in \mathbb{C}[\lambda]$ $\qquad$ $\chi_{\phi|_U}(\lambda) \in \underset{\mathcal{O}(U)}{\underset{\parallel}{\mathbb{C}[t]}}[\lambda]$

"The entries of ϕ are sections of ω_C"

$\mathrm{Tr}(\phi) \in H^0(C, \omega_C)$ $\qquad$ $\det \phi \in H^0(C, \omega_C^{\otimes n})$

$$\chi_\phi(\lambda) = \lambda^n + s_1 \lambda^{n-1} + \dots + s_n$$

$$s_i \in H^0(C, \omega_C^i)$$

<u>DEF</u> $$A_n := \bigoplus_{i=1}^{n} H^0(C, \omega_C^{\otimes i}) \simeq \mathbb{A}^N$$

<u>EX</u> USE R.R. to prove $N = n^2(g-1)+1$

The Hitchin ~~fibration~~ map

$$\chi : M_{n,d} \longrightarrow A_n$$
$$(E,\phi) \longmapsto (\text{coeff } \chi_E)$$

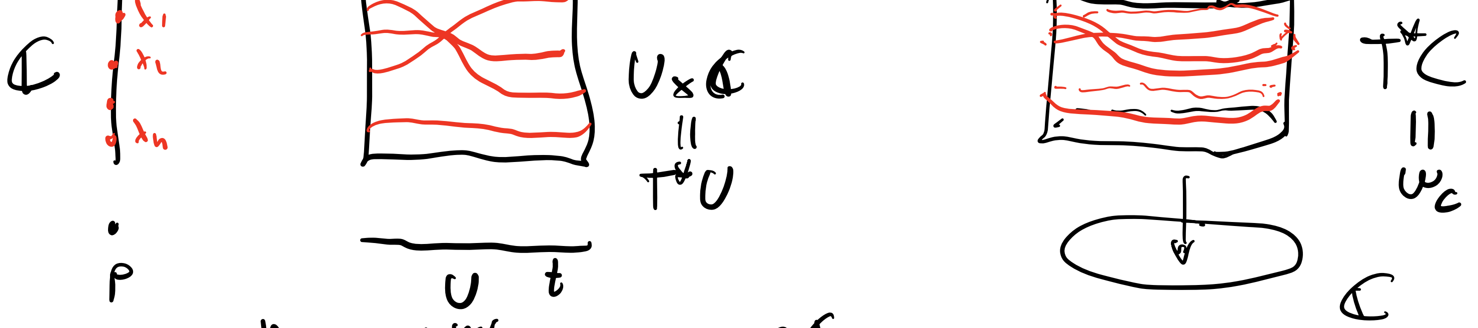
<u>REMARK</u>: $M_{n,d}$ is not projective, χ is surjective

- χ is a proper map

<u>Goal</u> $\chi^{-1}(a) = ? = \{(E,\phi) \mid \chi_\phi = q_a\}$

$a \in A_n$ $\qquad$ $q_a(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_n$

q_a can be evaluated at points of T^*C



$$q_a|_p = \lambda^n + (a_1)_p \lambda^{n-1} + \dots + (a_n)_p \in \mathbb{C}$$

$$q_a|_U = \lambda^n + a_1(t)\lambda^{n-1} + \dots + a_n(t)$$

$$q_a = \lambda^n + a_1 \lambda^{n-1} + \dots + a_n$$

$$s \in H^0(C, \omega_C)$$

<u>DEF</u> $C_a \subset T^*C$

$$\begin{tikzcd} C_a \arrow[d] & T^*C \arrow[dl] \\ C & \end{tikzcd}$$

C_a = Zero locus of $q_a(\lambda)$

very singular

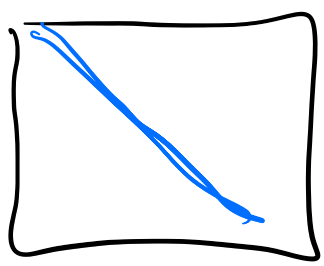
branched cover of degree n

C_a is called The spectral curve

<u>EX</u> $q_a = p_1^{k_1} \cdot p_2^{k_2} \dots p_r^{k_r}$

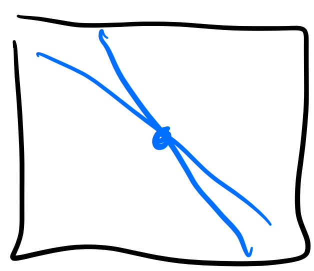
local

1 – $\lambda^2 + 2t\lambda + t^2 = (\lambda + t)^2$



T^*C

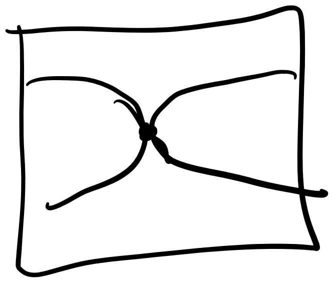
2 – $\lambda^2 + 3t\lambda + 2t^2 = (\lambda + t)(\lambda + 2t)$



3 - $\lambda^2 - t^3$

irreducible poly (curve)

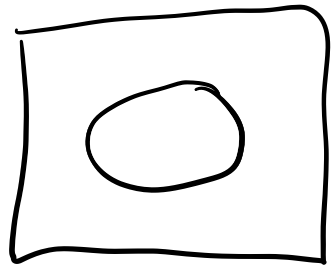
singular



4 - generic case

$\lambda^2 + t^2 - 1$

smooth and irreducible



$\chi^{-1}(a)$

$$M_{n,d} = \{ (E,\phi) \mid \begin{smallmatrix} rk & n \\ deg & d \end{smallmatrix} \}$$

$$\chi \downarrow$$

$$A_n \ni a$$

$$\begin{array}{c} C_a \subset T^*C \\ \downarrow \text{n-cover} \\ C \end{array}$$

<u>EX</u> C_a is smooth

$g(C_a) = n^2(g-1) + 1$

Hint 1 : adj formula & RR Thm

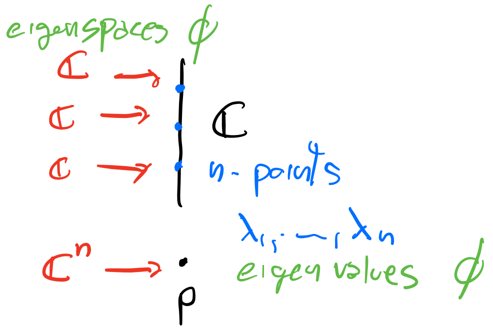
Hint 2 : R-H Thm & resultant

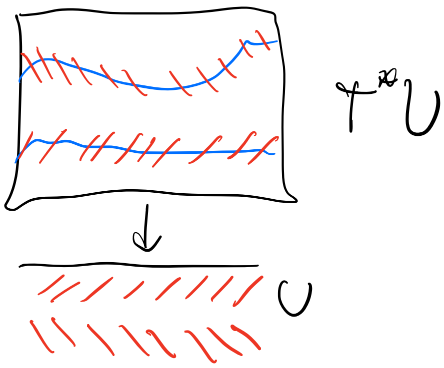
consider a s.t. C_a is smooth

L line bundle on C_a

$$\begin{array}{ccc} L & \longrightarrow & C_a \\ & & \downarrow \pi \\ \pi_* L & \rightsquigarrow & C \end{array}$$

↖ is a vector bundle





<u>FACT</u> . $\deg(\pi_* L) = \deg(L) - n(n-1)(g-1)$

$\Leftarrow \det \pi_* L = \operatorname{Norm}(L) \otimes \det \pi_* \mathcal{O}$

- There is a natural endomorphism on $\pi_* L$ is given by multiplication λ canonical section

$$\phi_p = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

<u>THM</u> Beauville Narasimhan Ramanan

if C_a is smooth Then

$$h^{-1}(a) \simeq J_{C_a}^{d+n(n-1)(g-1)} = \left\{ \begin{array}{l} \text{Line bundles on } C_a \\ \text{of degree } d+n(n-1)(g-1) \end{array} \right\}$$

$$(\pi_* L, \phi_L) \longleftarrow L$$

$$(E, \phi) \longmapsto \left((p, \lambda) \mapsto \text{gen eigenspace of } \underset{\text{associate } \lambda}{\phi_p} \right)$$

<u>COR</u> $\dim M_{n,d} = \dim A_n + \dim J_{C_a} = 2n^2(g-1) + 2$

Stratification of A_n

[illegible]

DEF $A_n^{red} \subset A_n$ $\quad A_n^{red} = \{ \underline{a} \mid q_{\underline{a}} = p_1^{k_1} \cdots p_r^{k_r},\ k_i = 1 \}$

$= \{ \underline{a} \mid C_{\underline{a}} \text{ is reduced} \}$

$$M_{n,d}^{red} := \chi^{-1}(A_n^{red})$$

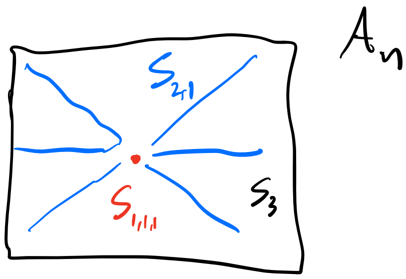
$$S_{\underline{n}} := \{ \underline{a} \mid q_{\underline{a}} = p_1 \cdots p_r \ \text{ s.t. } \deg p_i = n_i \}$$

$\underline{n} \vdash n \quad := \{ \underline{a} \mid C_{\underline{a}} \text{ has } r \text{ irr. components of degree } n_i \}$

$\|$

$\{n_1, n_2, \dots, n_r\}$

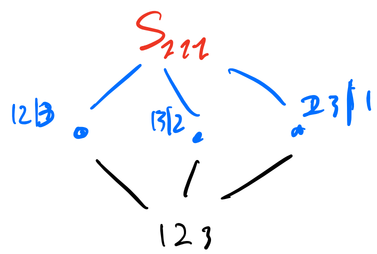
EG $n = 3$



$q_a = p_1 p_2 p_3 = r_1 r_2$

linear $\quad$ deg 2

branching around $S_{111} \longleftrightarrow$ set partition of $[3]$



$$A_{n_1} \times A_{n_2} \times \cdots \times A_{n_k} \longrightarrow S_{\underline{n}}$$

$$A_1 \times A_1 \times A_1 \longrightarrow S_{1,1,1}$$

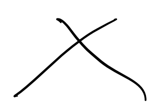
$$6 \longmapsto 1$$

Dual graph of $C_{\underline{a}}$

$\underline{a} \in S_{\underline{n}} \qquad \Gamma_{\underline{n}}$ on r vertices and x_{ij} edges $i - j$

$$x_{ij} := \# C_i \cap C_j = n_i n_j (2g-2) \in \mathbb{N}_+$$

$\{ \text{rank 1 Torsion free sheaves on } C_{\underline{a}} \}$



$L^{\vee}$

DEF L is $\underline{n}$-stable if

$$\chi(L|_{C'}) > \chi(L)\,\frac{\deg(C' \to C)}{n} \quad \forall\, C' \subsetneq C_a$$

$=n_i$

$$\chi(L) = \dim H^0(L) - \dim H^2(L) = \deg(L) - p_a + 1$$

PROP $x_i := \chi(L|_{C_i}) - (1-q)n_i$

L is $\underline{n}$-stable iff $\qquad \underline{n} = \{n_1 \ldots n_r\} \vdash n$

$$\sum_{i \in K} x_i > \sum_{i,j \in K} y_{ij} + \sum_{i \in K} \frac{d n_i}{n} \quad \forall\, K \subset [r],\ K \neq \emptyset$$

THM $\underline{a} \in S_{\underline{n}} \subseteq A_n^{red}$

$$J_{\underline{n}\text{-stable}}(C_a) = \left\{ \begin{matrix} \text{rank 1 T.F. sheaves} \\ \underline{n}\text{-stable of degree} \end{matrix} \right\} \xrightarrow{\sim} \chi^{-1}(\underline{a})$$

$$L \longmapsto (\pi_* L, \lambda \cdot)$$

n-stability $\longleftrightarrow$ μ-stability

has pure dimension $g(C_a) = n^2(g-1)+1$

$$\sum_{i \in [n]} x_i = \sum_{i,j \in [n]} y_{i,j} + d$$

Q # irr. components of $J(C_a) = ?$

are determined by $(x_1, \ldots, x_r)$

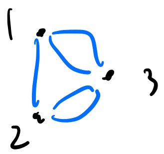
$$Z_{\underline{n},d} = \left\{ (x_1, \ldots, x_r) \in \mathbb{R}^r \;\middle|\; \begin{matrix} \sum x_i = \sum y_{ij} + d \\ \sum_{i \in K} x_i > \sum_{i,j \in K} y_{ij} + \sum_{i \in K} \frac{d n_i}{n} \end{matrix} \right\}$$

irr. components $= \#(Z_{\underline{n},d} \cap \mathbb{Z}^r) =: C(Z_{\underline{n},d})$

Cor # irr. cap $\chi^{-1}(a) = C(Z_{\underline{n},d})$

Combinatorics

Let Γ a graph on $[r]$

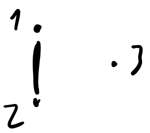
eg  $= \Gamma$

Let $T \vdash [r]$ $\quad T = \{T_1, T_2, \dots, T_k\}$

deleted graph $\Gamma_T = \left\{ e \in \Gamma \;\middle|\; \begin{array}{l} e : i - j \\ \exists\, k \text{ s.t. } i, j \in T_k \end{array} \right\}$

contracted graph $\Gamma^T = \Gamma / \Gamma_T \simeq \{ e \in \Gamma \setminus \Gamma_T \}$ on vertex set $[k]$

eg $T = 12|3$

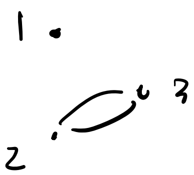
$\Gamma_T =$ 

Γ^T 

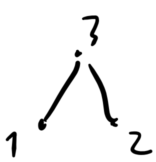
DEF a flat of Γ is $T \vdash [r]$ s.t.

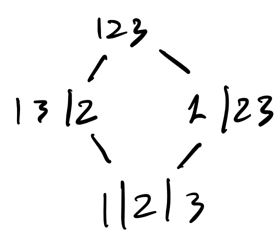
every block T_i induces a connected subgraph

The flats are ordered by refinement

eg $T = 1|23$ $\quad \Gamma_T =$ 

$\Gamma^T =$ 

eg $\Gamma =$ 

The flats are 

Graphical Zonotopes

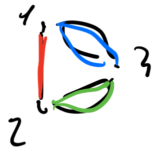
Γ be a graph

$$Z_\Gamma := \sum_{i,j} y_{ij} \underline{[e_i, e_j]} \subseteq \mathbb{R}^r$$

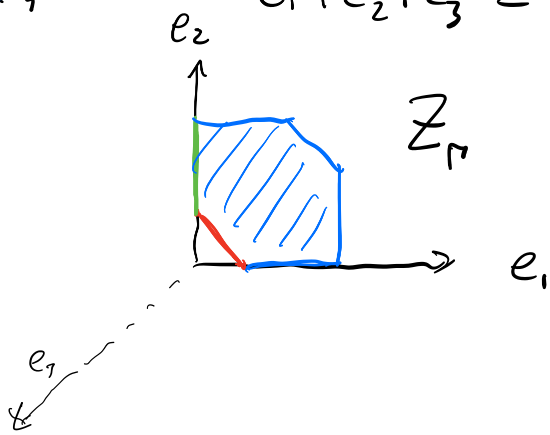
Minkowsky sum

\# edges between i and j

segment between e_i and e_j

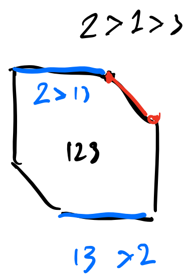
eg 

$e_1+e_2+e_3=5$



FACT • $Z_\Gamma = \left\{ (x_1 \dots x_r) \;\middle|\; \begin{array}{l} \sum x_i = \sum y_{ij} \\ \forall K \subset [r],\ K \neq \emptyset \quad \sum_{i \in K} x_i \geq \sum_{i,j \in K} y_{ij} \end{array} \right\}$

• $\{\text{faces of } Z_\Gamma\} \longleftrightarrow \left\{ (T, a) \;\middle|\; \begin{array}{l} T \text{ is a flat of } \Gamma \\ a \text{ is an acyclic orientation of } \Gamma^T \end{array} \right\}$

eg 

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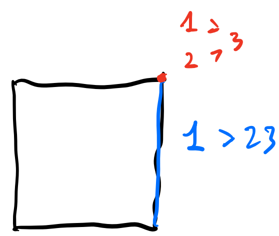


remark $\# \text{ int cap } \tau^{-1}(a) = C(Z_{\Gamma_n} + w)$

Γ_n is the dual graph of C_a $\quad w = (w_i) \quad w_i = \frac{d n_i}{n}$

Ehrhart Theory

DEF $\quad \operatorname{erh}_P(t) := \#(tP \cap \mathbb{Z}^r) \qquad t \in \mathbb{N}_+$

Eg $\quad P = [0, \tfrac{1}{2}] \subseteq \mathbb{R}$

$$\operatorname{ehr}_P(t) = \begin{cases} \frac{t}{2}+1 & \text{if } t \equiv 0 \ (2) \\ \frac{t+1}{2} & \text{if } t \equiv 1 \ (2) \end{cases}$$

Thm [Ehrhart '62, Macdonald '71]

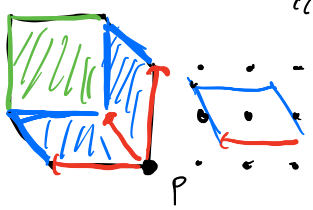
- if P has integral vertices then $\mathrm{ehr}_P(t)$ is a polynomial in t
- if P has rational vertices then $\mathrm{ehr}_P(t)$ is a quasi-polynomial
- if P is rational then

$$\mathrm{ehr}_P(-t) = (-1)^r \, \mathrm{ehr}_{\mathrm{Int}(P)}(t)$$

eg $P = (0, \tfrac{1}{2}) \in \mathbb{R}$

$$\mathrm{ehr}_P(t) = \begin{cases} \frac{t}{2} - 1 & \text{if } t \equiv 0 \ (2) \\ \frac{t-1}{2} & \text{if } t \equiv 1 \ (2) \end{cases}$$

FACT "nice partition of Z_P into parallelepipeds"



$[0, 2e_1) \times [0, 2e_2) + e_2$

$$Z_P = Z_\phi \sqcup Z_1 \sqcup Z_2 \sqcup Z_3 \sqcup Z_{12} \sqcup Z_{13} \sqcup Z_{23}$$

$$\mathrm{ehr}_P(t) = \mathrm{vol}(P)\, t^{\dim P} \quad \text{if } P \text{ is an alph open parallelepipeds}$$

ThM [Stanley '91, Ardila-Beck-McWhirter '20]

Γ is a graph $\omega \in \mathbb{R}^r$

$$\mathrm{ehr}_{Z_\Gamma + \omega}(t) = \sum_{\substack{I \text{ forest in } \Gamma \\ \mathrm{Aff\,Span}(Z_{\Gamma(I)} + t\omega) \cap \mathbb{Z}^r \neq \phi}} \mathrm{Vol}(I)\, t^{|I|}$$

DEF a flat $T = \{T_1, T_2, \ldots, T_k\} \vdash [r]$ is ω-integral

$t = -1$ if $\forall i$ $\sum_{j \in T_i} \omega_j \in \mathbb{Z}$ $\quad \omega_i \in \mathbb{R}$

Cor $$C(Z_\Gamma + w) = \sum_{\substack{T \text{ } w\text{-integral} \\ \text{flat}}} (-1)^{r - \ell(T)} \sum_{\substack{F \text{ spanning} \\ \text{forest of } \Gamma_T}} \mathrm{Vol}(F)$$

Ex compute $\#(\chi^{-1}(a))$ for $a \in A_n^{red}$:

- $n=4$ $d=0$ $g=2$
- $n=4$ $d=2$ $g=2$

Tomorrow $IH^\cdot(M_{n,d})$ χ $\begin{array}{c} M_{n,d}^{red} \\ \downarrow \\ A_n^{red} \end{array}$

$\chi^{-1}(a)$

$\#$ irr comp $\chi^{-1}(a) = C(Z_{\Gamma_{\underline{n}}} + w)$ $w_i = \frac{dn_i}{n}$

Intersection Cohomology

Q $IH^\cdot(M_{n,d}) = ?$

complex of sheaves

$$IH^\cdot(M_{n,d};\mathbb{Q}) := H^\cdot(M_{n,d}, \mathcal{IC})$$
$$= H^\cdot(A_n, R\chi_* \mathcal{IC})$$

$R\chi_* \mathcal{IC}\big|_{A_n^{red}}$

Theorem [Ngô, Maulik-Migliorini 2022]

$R\chi_* \mathcal{IC}|_{A^{red}} \simeq \bigoplus \mathcal{IC}_S(\mathcal{L}_{n,d} \otimes \Lambda_{\underline{n}})$

$A_n \quad \underline{n} \vdash n \qquad S_{\underline{n}}$

where • $\Lambda_{\underline{n}}$ is the cohomology sheaf of the relative Picard group $\mathrm{Pic}^0(\overline{C_{\underline{n}}})$ of the normalization of $C_{\underline{n}}$

• $\mathcal{L}_{\underline{n},d}$ unknown coefficients
← local system on $S_{\underline{n}}$

$\underline{Q1}$ $\underline{n}, d \qquad \mathcal{L}_{\underline{n},d} = 0$?

$\underline{Q2}$ $\operatorname{rk} \mathcal{L}_{\underline{n},d} = ?$

$\underline{Q3}$ monodromy action

$\underline{\text{Remark}}$ $\underline{a} \in S_{\underline{n}}$

$$\dim \mathcal{H}^{top}(R\chi_* IC)_{\underline{a}} = \# \text{ irr. comp of } \chi^{-1}(\underline{a})$$
$$= C(Z_{\Gamma_{\underline{n}}} + w)$$

$\underline{\text{LEMMA}}$ $a \in S_{\underline{n}} \quad \underline{n} = (n_1, \dots, n_r)$

$$\underset{\underset{n,d}{\uparrow}}{\mathcal{H}^{top}(R\chi_* IC)_{\underline{a}}} \simeq \bigoplus_{\underline{I} \vdash [r]} (\mathcal{L}_{n_{\underline{I}},d})_a \otimes \bigotimes_{i=1}^{\ell(t)} \mathcal{H}^{top}(R\chi_{\underset{\underset{n(t_i),d}{\uparrow}}{n(t_i),0}} IC)_{\underline{a}}$$

$\underline{EX}$ $\underline{n} = \{1,1,2,5\} \vdash 9 \qquad r=4 \qquad n=9$

$\underline{I} = \{13 \mid 24\} \vdash [4]$

$n(13) = 1+2 = 3 \qquad n(24) = 1+5 = 6$

$$(\mathcal{L}_{\{3,6\},d})_a \otimes \mathcal{H}^{top}(R\chi_{3,0} IC)_{\{1,2\}} \otimes \mathcal{H}^{top}(R\chi_{6,0} IC)_{\{\dots\}}$$

$\{1,5\}$

$$\subseteq \mathcal{H}^{Top}(R\mathcal{X}_{g,d}\ \mathcal{IC})_{\{1,1,2,3\}}$$

COR $\quad C(Z_{\Gamma_{\underline{n}}}+w) = \sum_{T\vdash[r]} rk(\mathcal{L}_{n_T,d})\, C(Z_{\Gamma_T,0})$

$\forall\ \underline{n} \vdash n$

restricted graph on the flat T

solve this recurrence relation

$(n)\vdash n \qquad C(Z_{\Gamma_n}+w) = \underset{\substack{\| \\ 1}}{rk(\mathcal{L}_{(n),d})}\, C(Z_{\Gamma_n})$

$w \in \mathbb{Z}$
$\quad\|$
$\frac{dn}{n}$

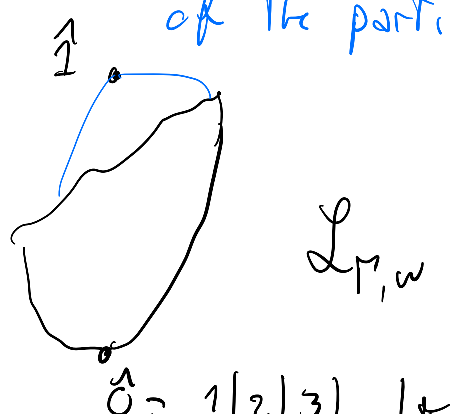
$\underline{n} = (a, n-a) \qquad (n) \sim rk(\mathcal{L}_{(a,n-a),d})$

THM [Mauri, Migliorini, P. 2023] $\quad\Gamma$ any graph $\quad w \in \mathbb{R}^r$

$$C(Z_\Gamma+w) = \sum_{\underline{S}\vdash[r]} \Big(\sum_{\underline{T}\geq \underline{S}} \mu_{\mathcal{L}}(S,T)\Big)\, C(Z_{\Gamma_S})$$

COR $\quad rk(\mathcal{L}_{\underline{n},d}) = \sum_{\substack{T\vdash[r] \\ T \text{ is } w\text{-integral}}} \underbrace{(-1)^{\ell(T)} \prod_{i=1}^{\ell(T)} (|T_i|-1)!}_{\mu \text{ mobius function of the partition lattice}}$

$\mathcal{L}_\Gamma$



$\mathcal{L}_{\Gamma,w}$

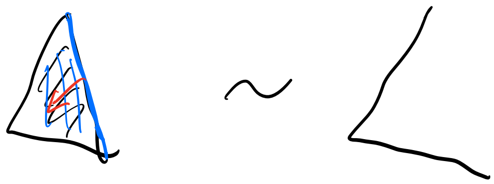
$$\sum_{T \text{ } \omega\text{-integral}} \mu(\hat 0, T) = -\sum_{T \text{ non } \omega\text{-integral}} \mu(\hat 0, T) = \mu_{\mathcal{L}_{\Gamma,\omega}}(\hat 0,\hat 1)$$

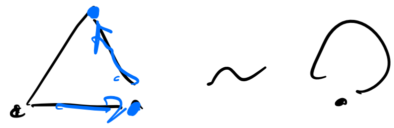
$$\mathrm{rk}\, \mathcal{L}_{\underline{n},d} = \mu_{\Gamma_{\underline n},\omega}(\hat 0,\hat 1) = \tilde\chi(\Delta(\mathcal{L}_{\Gamma_{\underline n},\omega}))$$

<u>THM</u> Hall's Theorem

$$\mu_{\mathcal L}(\hat 0,\hat 1) = \tilde\chi(\Delta(\mathcal L \setminus \{\hat 0,\hat 1\}))$$

Shellability





LEX - shellability

"good" matching on chains in $\mathcal L$" ⇝ prescribe a "good" collapsing on $\Delta(\mathcal L \setminus \{\hat 0,\hat 1\})$

<u>PROP</u>

If $\mathcal L$ is LEX-shellable $\Rightarrow$ $\Delta(\mathcal L \setminus \{0,1\}) \sim \vee S^{r-3}$

$$\mu_{\mathcal L}(\hat 0,\hat 1) = (-1)^{r-3}\, \mathrm{rk}\, H^{r-3}(\Delta(\mathcal L \setminus \{0,1\}))$$

$$\parallel$$

if $\underline n = (n) =$ or $\exists i$ $\frac{d n_i}{n} \notin \mathbb{Z}$ $\Rightarrow$ $\mathrm{rk}(\mathcal L_{\underline n,d}) > 0$

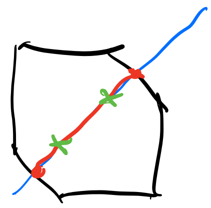
<u>THM</u> [Mauri, M, P]

$$(\mathcal L_{\underline n,d})_p \simeq H^{r-3}(\Delta(\mathcal L_{\underline n,\omega} \setminus \{\hat 0,\hat 1\})) \otimes \mathrm{sgn}$$

as repr. of $\pi_1(S_{\underline{n}}) \twoheadrightarrow S_{k_1} \times S_{k_2} \times \dots \times S_{k_n}$

$$\underline{n} = (1^{k_1}, 2^{k_2}, \underbrace{\dots}_{\sigma}, n^{k_n})$$

<u>IDEa</u> compute characters



$$M^{red}_{n,d}$$