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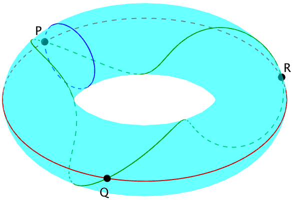
Combinatorics and Cohomology Algebra of Toric Arrangements

Matroids, Reflection Groups, and Free Hyperplane Arrangements



at RIMS, Kyoto University, June 12, 2018

Covered topics:

1. Cohomology Algebra
2. Combinatorics
3. Integer coefficients
4. Examples

Definitions

A *toric arrangement* $\mathcal{A}$ in the torus $T \simeq (\mathbb{C}^*)^r$ is a finite collection of (translates of) hypertori $\{D_e\}_{e \in E}$. Let $\Lambda := \mathrm{Hom}(T, \mathbb{C}^*) \simeq \mathbb{Z}^r$ be the character group of T and $\chi_e \in \Lambda$ a character defining D_e.

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In coordinates: each character is $\chi_e(t_1, \dots, t_r) = t_1^{a_1} t_2^{a_2} \cdots t_r^{a_r}$ and the hypertorus is

$$D = \{(t_1, \dots, t_r) \in (\mathbb{C}^*)^r \mid t_1^{a_1} t_2^{a_2} \cdots t_r^{a_r} = b\}.$$

The equations $\chi(\mathbf{t}) = b$ and $(-\chi)(\mathbf{t}) = b^{-1}$ define the same hypertorus.

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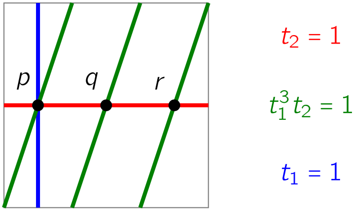
We want to study the cohomology algebra of the complement $M(\mathcal{A}) = T \setminus \bigcup_{e \in E} D_e$.



Generators

The cohomology of the torus is $H^1(T;\mathbb{Z}) = \{\mathrm{d}\log t_1^{a_1}\dots t_r^{a_r}\}_{\underline{a}\in\mathbb{Z}^r} \simeq \mathbb{Z}^r$ and $H^\bullet(T) = \bigwedge^\bullet H^1(T)$. We define $\psi_e = \mathrm{d}\log t_1^{a_1}\dots t_r^{a_r} \in H^1(T)$.

Remark

The form $\psi_B := \psi_{b_1}\cdots\psi_{b_k} \in H^{|B|}(T)$ is non-zero if and only if $\chi_{b_1},\dots,\chi_{b_k}$ are linearly independent.

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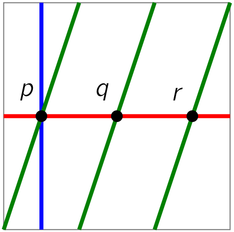

We define
$\omega_e = \mathrm{d}\log(b - t_1^{a_1}\cdots t_n^{a_n}) + \mathrm{d}\log(b^{-1} - t_1^{-a_1}\cdots t_n^{-a_n})$.
Observe that

$$\omega_1 \cdot \omega_2 = \omega_{p,1,2} + \omega_{q,1,2} + \omega_{r,1,2};$$

these two-forms are linearly independent.

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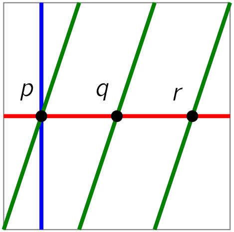

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Observe that

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these two-forms are linearly independent.

In general we define the differential forms $\omega_{W,A}$ for each independent set $A \subset E$ and W connected component of $\cap_{i\in A} D_i$.

Relations

If $A_1 \sqcup A_2$ is dependent then $\omega_{W_1,A_1}\omega_{W_2,A_2} = 0$, otherwise

$$\omega_{W_1,A_1}\omega_{W_2,A_2} = \pm \sum_{L \text{ c.c. } W_1 \cap W_2} \omega_{L,A_1 \sqcup A_2}. \tag{1}$$

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Finally, the following non trivial relation holds for every circuit C and c.c. L of $\cap_{i\in C} D_i$

$$\sum_{j\in C} \sum_{\substack{A \sqcup B \sqcup \{j\} = C \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} \frac{m(A)}{m(A \cup B)} \omega_{W,A} e_B \psi_B = 0, \tag{3}$$

where $m(A')$ is the number of c.c. of $\cap_{i\in A'} D_i$, W is the connected component containing L and $e_B = \prod_{i\in B} \operatorname{sgn} n_i$ for $\sum_{i\in C} n_i \chi_i = 0$.

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Remark

The numbers $\operatorname{sgn} n_i$ correspond to the choice of an orientation for the toric arrangement.

The cohomology algebra

Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. – June '18)

The rational cohomology algebra of the complement $M(\mathcal{A}) \subset T$ is generated by $H^1(T)$ and by $\omega_{W,A}$, for A independent and W c. c. of $\cap_{i\in A} D_i$, with relations

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This theorem is a generalization of the result in the unimodular case by De Concini and Procesi (2005).

Question: How does the cohomology ring depend on the combinatorics?

Combinatorial objects

Equations

From now on we suppose all arrangements to be central, i.e.

$$D_i = \{\underline{t} \in (\mathbb{C}^*)^r \mid t_1^{a_{1,i}} \dots t_r^{a_{r,i}} = 1\}.$$

We collect these data in a matrix with integer coefficients $N = (a_{i,j}) \in M(r, n; \mathbb{Z})$.

Example: The equations

$$x = 1$$

$$y = 1$$

$$xy^3 = 1$$

are described by the matrix

$$N = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Combinatorial objects

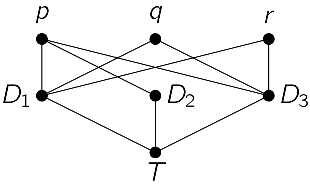
Equations

⇝

Poset of layers

The poset of layers $\mathcal{L}(\mathcal{A})$ is the set of connected components of intersections ordered by reverse inclusion.

Example: the Hasse diagram of the poset of layers is



Every interval of $\mathcal{L}(\mathcal{A})$ is a geometric lattice ranked with the codimension in T.

Combinatorial objects

Equations

$\rightsquigarrow$

Poset of layers

$\rightsquigarrow$

Arithmetic matroid

Definition: An *arithmetic matroid* is a ground set E with the rank function rk and the multiplicity function m.

Example: The ground set is $E = [n]$, the set of hypertori. The rank function is $\mathrm{rk}(A) = \mathrm{codim}_T(\cap_{i\in A} D_i)$ and the multiplicity function is $m(A) = \#$ c.c. of $\cap_{i\in A} D_i$.

Combinatorial objects

Equations

$\rightsquigarrow$

Poset of layers

$\rightsquigarrow$

Arithmetic matroid

$\rightsquigarrow$

Arithmetic Tutte polynomial

The *arithmetic Tutte polynomial* of an arithmetic matroid is

$$T(x,y) := \sum_{A \subseteq E} m(A)(x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}$$

Theorem (Moci – 2012)

The Poincaré polynomial of $M(\mathcal{A})$ is

$$P(q) = q^{\mathrm{rk}(E)}\, T\left(\frac{2q+1}{q}, 0\right).$$

Theorem (d'Antonio, Delucchi – 2013)

The cohomology with integer coefficients of $M(\mathcal{A})$ is torsion free.

Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. – June '18)

The rational cohomology algebra of the complement $M(\mathcal{A}) \subset T$ *is generated by* $H^1(T)$ *and by* $\omega_{W,A}$*, for* A *independent and* W *c. c. of* $\cap_{i \in A} D_i$*, with relations*

$$\omega_{W_1,A_1}\omega_{W_2,A_2} = \pm \sum_{L \text{ c.c. } W_1 \cap W_2} \omega_{L, A_1 \sqcup A_2} \tag{1}$$

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where $e_i = \operatorname{sgn} n_i$ *if* $\sum_{i \in C} n_i \chi_i = 0$.

Representation of arithmetic matroids

$$
\begin{tikzcd}
\mathsf{GL}_r(\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^n \arrow[d, hook] & \circlearrowright & M(r,n;\mathbb{Z}) \arrow[d, hook] \\
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Theorem (P. – 2017)

Suppose that N_1 and $N_2 \in M(r,n;\mathbb{Z})$ are two representation of the arithmetic matroid (E, rk, m) with $m(\varnothing) = 1$. Then there exists $g \in \mathrm{GL}_r(\mathbb{Q}) \times (\mathbb{Z}/2\mathbb{Z})^n$ such that $N_2 = gN_1$.

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Corollary (P. – 2017)

An arithmetic matroid (E, rk, m) *with* $m(\varnothing) = m(E) = 1$ *has at most one essential representation (up to equivalence).*

Orientable arithmetic matroids

Definition

An *oriented arithmetic matroid* is a matroid (E, rk) with two extra data: a orientation χ and a multiplicity function m such that

$$\sum_{i=0}^{r}(-1)^i \chi m(y_i, x_2, \dots, x_r)\chi m(y_0, \dots, \hat{y}_i, \dots, y_r) = 0. \qquad \text{(GP)}$$

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Theorem (P. – 2018)

If $(E, \mathrm{rk}, m, \chi)$ *and* $(E, \mathrm{rk}, m, \chi')$ *are two oriented arithmetic matroids then* χ' *is a reorientation of* χ.

Thus we call these triples (E, rk, m) *orientable arithmetic matroids*. Moreover in the realizable case we have:

$$\sum_{i \in C} \chi(c_0, \dots, \hat{c}_i, \dots, c_r) m(C \setminus \{i\})\psi_i = 0 \quad \in H^1(T)$$

so that $e_i = \chi(c_0, \dots, \hat{c}_i, \dots, c_r)$.

Let $\mathcal{A}$ be an essential arrangement.

Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. – June '18)

The rational cohomology algebra of the complement $M(\mathcal{A}) \subset T$ is generated by $\{\psi_i\}_{i \leq n}$ and by $\omega_{W,A}$, for A independent and W c. c. of $\cap_{i \in A} D_i$, with relations

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where $e_B = \prod_{i \in B} \chi(c_0, \ldots, \hat{c}_i, \ldots, c_k)$.

This presentation depends only on the poset of layers $\mathcal{L}(\mathcal{A})$.

Presentation with integer coefficients

Define the forms $\epsilon_i = \mathrm{d}\log(b - t_1^{a_1} \cdots t_n^{a_n})$ and from these we define the forms $\epsilon_{W,A}$ for every A independent set and W c.c. of $\cap_{i\in A} D_i$.
The following holds:

$$\omega_{W,A} = \sum_{B \subseteq A} (-1)^{|B|} 2^{|A\setminus B|} \frac{m(A \setminus B)}{m(A)} \epsilon_{L, A\setminus B} \psi_B,$$

where L is the c.c. of $\cap_{i \in A\setminus B}$ containing W.

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Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. – June '18)

The integral cohomology algebra of $M(\mathcal{A})$ is generated by $H^1(T;\mathbb{Z})$ and $\epsilon_{W,A}$, with relations

$$\epsilon_{W_1,A_1} \epsilon_{W_2,A_2} = \pm \sum_{L \text{ c.c. } W_1\cap W_2} \epsilon_{L,A_1\sqcup A_2}$$

$$\epsilon_{W,A}\psi = 0 \quad \text{if} \quad \psi_{|W} = 0$$

$$\frac{1}{2^{|C|-1}} \sum_{j\in C} \sum_{\substack{A\sqcup B\sqcup \{j\}=C \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} \frac{m(A)}{m(A\cup B)} \omega_{W,A} e_B \psi_B = 0.$$

First example

Consider the two central arrangements described by the matrices

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 7 & 7 \end{pmatrix} \qquad \qquad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 7 \end{pmatrix}.$$

The two arrangements have the isomorphic poset of layers and therefore same arithmetic matroid.

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Consider the two central arrangements described by the matrices

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 7 & 7 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{7} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 7 \end{pmatrix}.$$

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However, the cohomology algebras with rational coefficients are isomorphic.

Second example

Consider the two central arrangements described by the matrices

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 4 & 1 & 6 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix}.$$

The two arrangements have the same associated arithmetic matroid (the same matroid over $\mathbb{Z}$) but different poset of layers.

Second example

Consider the two central arrangements described by the matrices

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{5} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 & 6 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix}.$$

The two arrangements have the same associated arithmetic matroid (the same matroid over $\mathbb{Z}$) but different poset of layers.

The two cohomology algebra with rational coefficient are not isomorphic.

Further developments

1. Defining and studying "pseudo-toric arrangements".
2. Defining a "good" class of poset containing all poset of layers.
3. Working with other generalizations of matroids (e.g. G-semimatroids).
4. Studying toric resonance varieties.

Thanks for listening!