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Chow ring of polymatroids

joint work with Gian Marco Pezzoli

MIT-Harvard-MSR Combinatorics Seminar

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Covered topics:

Characteristic polynomial

Combinatorial Hodge theory

Polymatroids

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$$P_G(k) = P_{G\setminus e}(k) - P_{G/e}(k)$$

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for any edge e and proceed by induction on the number of edges. $\square$

Notation: the *characteristic polynomial* is $p_G(k) = P_G(k)/k^{\# cc\, G}$.

The characteristic polynomial of the graph G is:

$$p_G(x) = \omega_0 x^r + \omega_2 x^{r-1} + \cdots + \omega_r$$

Conjecture (Read '68)

The sequence ω_i is *unimodular*:

$$\omega_0 \leq \omega_1 \leq \cdots \leq \omega_k \geq \cdots \geq \omega_{r-1} \geq \omega_r.$$

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Conjecture (Hoggar '74)

The sequence ω_i is *log-concave*:

$$\omega_i^2 \geq \omega_{i-1}\omega_{i+1}.$$

Matroids

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1. hyperplanes arrangements,
2. cycles of a graph,
3. linear dependencies among vectors.

There are a lot of equivalent definition:

1. rank function,
2. bases, independent sets, circuits,
3. geometric lattices,
4. integral polytopes.

Definition

A *matroid* M is a pair $(E, \mathrm{rk}\colon 2^E \to \mathbb{N})$ such that:

1. $\mathrm{rk}(A) \leq |A|$ for all $A \subseteq E$,
2. (increasing) $\mathrm{rk}(A) \leq \mathrm{rk}(B)$ for all $A \subseteq B \subseteq E$,
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For $G = (V, E)$ we define the *cycle matroid* $M(G) = (E, \mathrm{rk})$ where $\mathrm{rk}(A) = |V_A| - \# cc\, A$. Moreover $p_{M(G)} = p_G$.

The characteristic polynomial of the matroid M is:

$$p_M(x) = \omega_0 x^r + \omega_2 x^{r-1} + \cdots + \omega_r.$$

Conjecture (Rota '71, Heron '72)

The sequence ω_i is *unimodular*:

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Definitions

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- the element $\ell \in A^1$ satisfies the *Hodge Riemann relations* if
$$Q_\ell^k\colon A^k \times A^k \to \mathbb{Q}$$
defined by $Q_\ell^k(a, b) = (-1)^k \deg(a\ell^{n-2k} b)$ (for $k \leq \frac{n}{2}$) is positive defined on the subspace
$$P_k = \ker(\cdot\ell^{n-2k+1}\colon A^k \to A^{n-k+1}).$$

Theorem

If X is a compact manifold then $H(X)$ satisfies Poincaré duality. Moreover if X is a compact Kahler manifold with Kahler class ω then ω satisfies Hard Lefschetz and Hodge Riemann.

More generally, any ample class $\ell \in H^2(X)$ satisfies Hard Lefschetz and Hodge Riemann.

Theorem (Adiprasito, Huh, Katz '18)

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They define a Chow ring $A(M)$ and two classes $\alpha, \beta \in A^1(M)$ such that $\deg(\alpha^{r-k}\beta^k) = \omega_k$.

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$\deg(\alpha\beta^{r-1})^2 \geq \deg(\alpha^2\beta^{r-2})\deg(\beta^r)$ and to

$$\det \begin{pmatrix} Q_\beta(\alpha,\alpha) & Q_\beta(\beta,\alpha) \\ Q_\beta(\alpha,\beta) & Q_\beta(\beta,\beta) \end{pmatrix} \leq 0.$$

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But Q_β restricted to $\langle \alpha, \beta \rangle$ has signature $(1,1,0)$. $\square$

Polymatroids

A *polymatroid* P is a pair $(E, \mathrm{cd}\colon 2^E \to \mathbb{N})$ such that

1. $\mathrm{cd}(\emptyset) = 0$,
2. (increasing) $\mathrm{cd}(A) \leq \mathrm{cd}(B)$ for all $A \subseteq B \subseteq E$,
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There are equivalent definition in term of independent sets, bases, generalized permutahedra.
Polymatroids codify the combinatorics of:

1. subspace arrangements,
2. cycles in an hypergraph.

A *k-flat* $F \subseteq E$ is a maximal subset such that $\mathrm{cd}(F) = k$.

The poset of flats

Definition (Poset of flats)

Let $L(P)$ be the set of all flats of the polymatroid P ordered by reverse inclusion.

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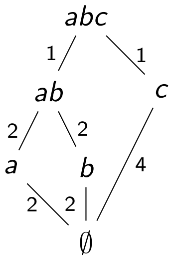
Example



In general $L(P)$ is not a geometric lattice and is not ranked.

Building sets

A subset $\mathcal{G} \subset L$ is a *building set* if for all $x \in L$

$$[\hat{0}, x] = \prod_{y \in \max(\mathcal{G}_{\leq x})} [\hat{0}, y]$$

and

$$\mathrm{cd}(x) = \sum_{y \in \max(\mathcal{G}_{\leq x})} \mathrm{cd}(y).$$

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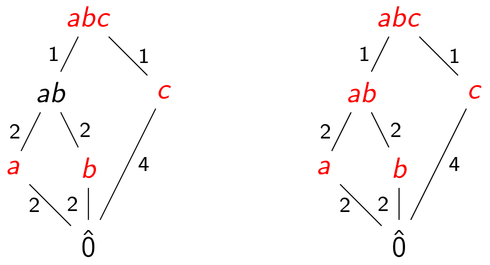

Previous works

- De Concini, Procesi '95 described the Chow ring $A(Y_{\mathcal{A},\mathcal{G}})$ (cohomology) of wonderful models.
- Feichtner, Yuzvinsky '03 described the Chow ring $A(L, \mathcal{G})$ of an atomic lattice with a building set.
- Huh, Adiprasito, Katz '18 proved the Kähler package for $A(L)$ of a geometric lattice with the maximal building set.

Chow ring

Define the algebra $A(P, \mathcal{G})$ is generated by x_W for $W \in \mathcal{G}$ with relations:

$$\Big(\sum_{Z \geq W} x_Z\Big)^b \prod_{V \in S} x_V = 0$$

for $S \subset \mathcal{G}$, $W \in \mathcal{G}$ and $b = \mathrm{cd}(W) - \mathrm{cd}(\bigvee(S_{<W}))$.

Simplicial generation

We perform an upper triangular base change by defining $\sigma_W = \sum_{Z \geq W} x_Z$.

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The Chow ring $A(P, \mathcal{G})$ is generated by σ_W for $W \in \mathcal{G}$ with relations:

$$\sigma_W^b \prod_{V \in S} (\sigma_V - \sigma_W) = 0$$

for $S \subset \mathcal{G}$, $W \in \mathcal{G}$ and $b = \mathrm{cd}(W) - \mathrm{cd}(\bigvee(S_{<W}))$,

Let M be a matroid and $\mathcal{G} = L(M) \setminus \{\hat{0}\}$ be the maximal building set.

Theorem (Adiprasito, Huh, Katz '18)

The ring $A(M, L(M) \setminus \{\hat{0}\})$ is a Poincaré duality algebra and each $\ell = \sum_{W \neq \hat{1}} c_W x_W \in A^1(M, L(M) \setminus \{\hat{0}\})$ such that

$$c_W + c_Z > c_{W \cup Z} + c_{W \cap Z}$$

satisfies Hard Lefschetz and Hodge Riemann relations.

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Theorem (P. Pezzoli '21)

The ring $A(P, \mathcal{G})$ is a Poincaré duality algebra and each $\ell = -\sum_{W \in \mathcal{G}} d_W \sigma_W \in A^1(P, \mathcal{G})$ such that

$$d_W > 0$$

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satisfies Hard Lefschetz and Hodge Riemann relations.

We call this orthant the σ-cone.

Remark

The σ-cone is contained in the ample cone of any realization, but for polymatroids the ample cone depends on the chosen realization.

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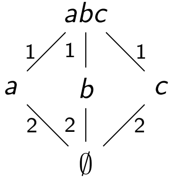
Example

Consider the polymatroid realized by three distinct lines in $\mathbb{C}^3$.



$Y_{\mathcal{G}}$ is the blowup of $\mathbb{P}^2$ in three points. If the three points are in general position then the ample cone coincides with the σ-cone.

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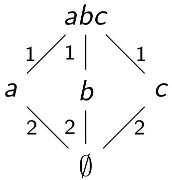

$Y_{\mathcal{G}}$ is the blowup of $\mathbb{P}^2$ in three points. If the three points are in general position then the ample cone coincides with the σ-cone.Otherwise the three points are collinear and the ample cone is given by:

$$\{-d_{abc}\sigma_{abc} - d_a\sigma_a - d_b\sigma_b - d_c\sigma_c \mid d_a, d_b, d_c > 0,$$
$$d_{abc} > -\min(d_a, d_b, d_c)\}$$

Remark

There are examples of polymatroids with (reduced) characteristic polynomial with negative coefficients and that do not form a log-concave sequence.

Main lemmas

We needed to compute $\mathrm{Ann}(x_W)$:

Lemma 1

For $W \neq \hat{1}$ there is an isomorphism

$$A(P,\mathcal{G}) \big/ \mathrm{Ann}(x_W) \cong A(P_W, \mathcal{G}_W) \otimes A(P^W, \mathcal{G}^W).$$

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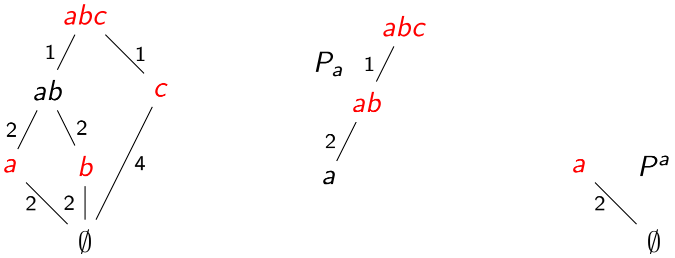
This looks like a Deletion-Restriction argument



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We needed to compute $\operatorname{Ann}(-\sigma_W)$:

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If $\operatorname{cd}(W) > 1$ *there is an isomorphism*

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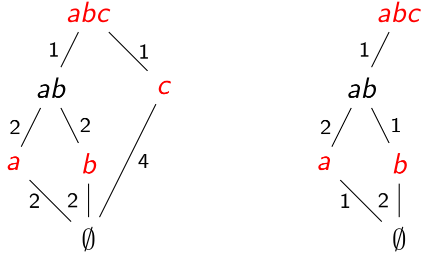
Idea: truncation at a consists in cutting the subspace arrangement with a generic hyperplane containing the flat a.



$\mathrm{tr}_a P$

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We needed to compute $\operatorname{Ann}((x_a - \sigma_a)^{\operatorname{cd}(a)})$ for an atom a:

Lemma 3

There is an isomorphism

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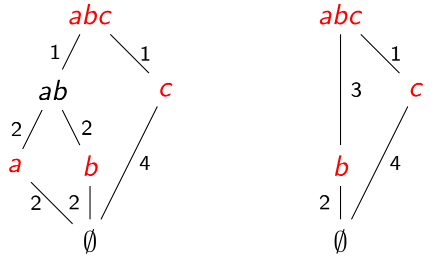
Idea: remove a but not the elements in $\mathcal{G} \setminus \{a\}$.



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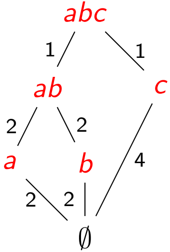

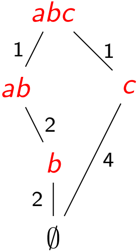

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1. Present a Gröbner basis for $A(P, \mathcal{G})$,
2. Prove Poincaré duality constructing an explicit pairing,
3. Prove the previous lemmas using Poincaré duality,
4. Prove simultaneously Hard Lefschetz and Hodge Riemann by induction on $|\mathcal{G}|$.

$\square$

1. Present a Gröbner basis for $A(P, \mathcal{G})$

Proposition (Feichtner Yuzvinsky '04, Bibby Denham Feichtner '20, P. Pezzoli '21)

The relations defining $A(P, \mathcal{G})$ form a Gröbner basis with respect the *deg-lex order*:

$$\Big(\sum_{Z \geq W} x_Z\Big)^b \prod_{V \in S} x_V = 0$$

for $S \subset \mathcal{G}$, $W \in \mathcal{G}$ and $b = \mathrm{cd}(W) - \mathrm{cd}(\bigvee(S_{<W}))$.

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for $S \subset \mathcal{G}$, $W \in \mathcal{G}$ and $b = \operatorname{cd}(W) - \operatorname{cd}(\bigvee(S_{<W}))$. Moreover, a additive basis of $A(P, \mathcal{G})$ is given by

$$\prod_{W \in S} x_W^{m_W}$$

where S is $\mathcal{G}$-*nested* and $m_W < \operatorname{cd}(W) - \operatorname{cd}(\bigvee(S_{<W}))$.

2. Prove Poincaré duality constructing an explicit pairing

Define a bijection ϵ from a linear basis of A^k to a linear basis of A^{r-k} such that

- $x_S^m \epsilon(x_S^m) = \pm 1$,
- $x_S^m \epsilon(x_T^n) = 0$ if $x_S^m \prec_{rev-lex} x_T^n$.

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Proof.

Indeed, the matrix representing the multiplication in the basis $\{x_S^m\}$ and $\{\epsilon(x_S^m)\}$ is upper triangular with diagonal entries ± 1. □

3. Prove the previous lemmas using Poincaré duality

Proposition

Let A, B be two Poincaré duality algebra of dimension r and $f \colon A \to B$ a surjective morphism. Then f is an isomorphism.

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$$A(P,\mathcal{G})/_{\mathrm{Ann}(x_W)} \cong A(P_W,\mathcal{G}_W) \otimes A(P^W,\mathcal{G}^W)$$

$$A(P,\mathcal{G})/_{\mathrm{Ann}(-\sigma_W)} \cong A(\mathrm{tr}_W P, \mathrm{tr}_W \mathcal{G})$$

$$A(P,\mathcal{G})/_{\mathrm{Ann}((x_a-\sigma_a)^{\mathrm{cd}(a)})} \cong A(P(a),\mathcal{G}(a))$$

4. Prove Hard Lefschetz and Hodge Riemann by induction

Proposition (Adiprasito Huh Katz '18)

If $\ell = -\sum_W c_W \sigma_W \in A^1$ with $c_W > 0$ such that $\bar{\ell}$ satisfies $HR(A/\operatorname{Ann}(-\sigma_W))$ for all W, then ℓ satisfies $HL(A)$.

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Proposition (Adiprasito Huh Katz '18)

Let C be a PD algebra, $p(x) = x^d + \mu_{d-1}x^{d-1} + \cdots + \mu_0 \in C[x]$ be a polynomial with $\mu_0 \neq 0$, $B = C/Ann(\mu_0)$ and $A = C[x]/(xAnn(\mu_0), p(x))$. If $\ell \in C^1$ satisfies $HR(C)$ and $HR(B)$, then $\ell + \epsilon x$ satisfies $HR(A)$ for sufficiently small $\epsilon > 0$.

Thanks for listening!

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