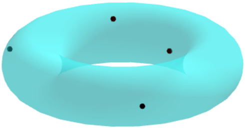
Roberto Pagaria

Scuola Normale Superiore

# Cohomology of configuration spaces of points on the 2-torus

InterCity - seminar



at University of Neuchâtel
November 30, 2018

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$$\mathcal{C}^n(E) := \{(p_1, \dots, p_n) \in E^n \mid p_i \neq p_j\}$$

$$\mathcal{UC}^n(E) := \{X \subset E \mid |X| = n\} \simeq \mathcal{C}^n(E)/S_n$$

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- It is open in the *Hilbert scheme*.
- It is related to the *motion planning problem*.
- It is an example of *elliptic arrangement*.
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**Our plan:**

- Leray spectral sequence for $\mathcal{C}^n(E) \hookrightarrow E^n$.
- Mixed Hodge theory for the degeneration of SS (Kriz model).
- Representation theory of $S_n$ to compute the model for $\mathcal{UC}^n(E)$.
- Some non-trivial computations (not shown).

# What is a spectral sequence?

It is a collection $(E_m, \mathrm{d}_m)_{m\in\mathbb{N}}$ of CDGA such that $E_{m+1} = H(E_m, \mathrm{d}_m)$.

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Figure: The bigraded algebra $E_2$ with differential $d_2$.

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Figure: The bigraded algebra $E_\infty$ .

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# The Leray SS

Let $j\colon X \to Y$ be a continuous map and $\mathcal{F}$ be a sheaf on $X$. Define the *higher direct image* sheaves $R^q j_* \mathcal{F}$ on $Y$ by

$$U \mapsto H^q(j^{-1}U, \mathcal{F}).$$

**Theorem (Leray '46)**

*There exists a SS* $(E_m, \mathrm{d}_m)$ *such that:*

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We apply this to $j : \mathcal{C}^n(E) \hookrightarrow E^n$ and $\mathcal{F} = \mathbb{Q}_{\mathcal{C}^n(E)}$.
In this case we have

$$R^q j_* \mathbb{Q}_{\mathcal{C}^n(E)} = \bigoplus_{\operatorname{codim} W = q} \mathbb{Q}_W \otimes H^q(\mathcal{C}^q(\mathbb{C})),$$

where $W \simeq E^{n-q}$.

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*For algebraic varieties there exists a functorial decomposition:*

$$H^k(X) = \bigoplus_{w=0}^{2\dim X} \bigoplus_{p+q=w} H^{p,q}(X).$$

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**Corollary**

*The map* $\mathrm{d}_m \colon E_m^{p,q} \to E_m^{p+m,q+1-m}$ *is zero for* $m > 2$.

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We describe $(E_2, \mathrm{d}_2)$ explicitly: $E_2$ is the external algebra on generators

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# Action of the symmetric group

The action of $\sigma \in S_n$ on $E_2$ is given by $\sigma x_i = x_{\sigma^{-1}(i)}$, $\sigma y_i = y_{\sigma^{-1}(i)}$, and $\sigma \omega_{i,j} = \omega_{\sigma^{-1}(i),\sigma^{-1}(j)}$.

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**Theorem (Ashraf, Azam, Berceanu ’12)**

*The Krǐz model decomposes as*

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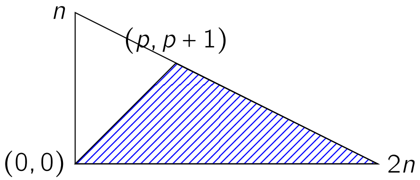


**Corollary (P. '18)**

*For $q > p+1$ we have* $(E_2^{p,q})^{S_n} = 0$

# The Betti numbers

Let $T_n(t)$ be the truncation at degree $n$ of

$$T(t) = \frac{1+t^3}{(1-t^2)^2} = 1 + 2t^2 + t^3 + 3t^4 + 2t^5 + 4t^6 \dots$$

**Theorem (Drummond-Cole, Knudsen '17, Maguire '16, Schiessl '16)**

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We need to study only $\mathcal{C}^n(E)/E \simeq \mathcal{C}^{n-1}(E \setminus p)$ that has Poincaré polynomial equal to $T_{n-1}(t)$.

The model $E_2$ for $\mathcal{C}^n(E)/E$ differs from that of $\mathcal{C}^n(E)$ by adding the relations $\sum_i x_i = 0$ and $\sum y_i = 0$.

# Action of the MPC

The mapping class group $\mathrm{MCG}(E) \sim SL_2(\mathbb{Z})$ acts on $\mathcal{C}^n(E)$ and therefore on $E_2$ as follows:

- $\omega_{i,j}$ are invariants.
- $\langle x_i, y_i \rangle$ is invariant and isomorphic to the irreducible representation $\mathbb{V}_1$.

This action extends to $SL_2(\mathbb{Q})$.

Recall that the irreducible representations of $SL_2(\mathbb{Q})$ are $\mathbb{V}_n = S^n\mathbb{V}_1$ of dimension $n+1$.

## Theorem (P. ’18)

*The cohomology* $H(E_2^{S_n}, \mathrm{d}_2)$, *for* $n = 2p$, *is:*

$$\begin{array}{c|cccccc}
p-1 & & & & & \mathbb{V}_{p-1} & \mathbb{V}_{p-2} \\
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**Sketch of proof.**

Consider the elements $\alpha \in E_2^{1,1}$ and $\beta \in E_2^{1,2}$ defined by:

$$\alpha := \sum_{i,k<h} (x_i - x_k)\omega_{k,h}$$

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The cohomology $H(E_2^{S_n}, \mathrm{d}_2)$ is generated as $SL_2(\mathbb{Q})$-module by $\alpha^k$ and $\alpha^k\beta$. □

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## Corollary (P. '18)

*The cohomology algebra $H(\mathcal{UC}^n(E))$ is given by $H(E) \otimes S^\bullet\mathbb{V}_1[\beta]$ with relations*

- *for $n = 2p$: $\alpha^p = \alpha^{p-1}\beta = \beta^2 = 0$.*
- *for $n = 2p+1$: $\alpha^{p+1} = \alpha^{p-1}\beta = \beta^2 = 0$.*

*Moreover, $\mathcal{UC}^n(E)$ is a formal space.*

# Graphic elliptic arrangements

Let $\mathcal{G} = ([n], \mathcal{E})$ be a graph. Define

$$M_{\mathcal{G}} := \{(p_1, \dots, p_n) \in E^n \mid p_i \neq p_j \text{ for } (i,j) \in \mathcal{E}\}.$$

Consider the Leray SS associated with $M_{\mathcal{G}} \hookrightarrow E^n$; we have $E_3(M_{\mathcal{G}}) = E_\infty(M_{\mathcal{G}}) = H(M_{\mathcal{G}})$.

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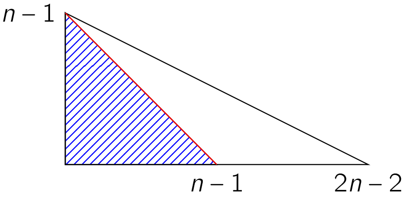
As done before we add the relations $\sum_i x_i = \sum_i y_i = 0$. The third page $E_3(M_{\mathcal{G}})$ is non-zero only when $p + q < n$,

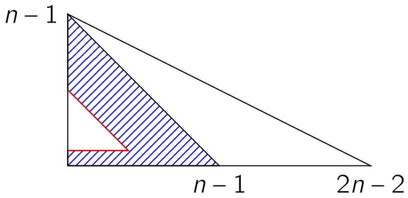



because $M_{\mathcal{G}}/E$ has the homotopy type of a CW-complex of dimension $n-1$.

Suppose now that $\mathcal{G}$ has no cycles (circuits) of length $\leq k$.

## Conjecture

*The groups* $E_3^{p,q}(M_{\mathcal{G}})$ *are zero for* $p+q<k$ *and* $q>0$*.*



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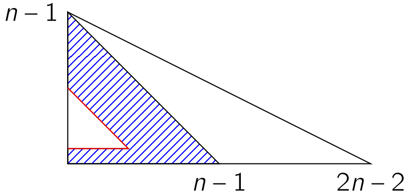
**Conjecture**

*The groups $E_3^{p,q}(M_{\mathcal{G}})$ are zero for $p+q<k$ and $q>0$.*




This conjecture is equivalent to one of the following:

- $H^i(E^n \setminus M_{\mathcal{G}})$ has pure mixed Hodge structure for $i > 2n-k$.
- computing the dimension of $\Lambda^\bullet(x_i, y_i)/((x_i - x_j)(y_i - y_j))_{(i,j)\in\mathcal{E}}$ in degree less than $k+2$.

# Thanks for listening!

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