Roberto Pagaria
Università di Bologna

Representation theory and configuration spaces

Seminar on Combinatorics, Lie Theory, and Topology

December 9, 2020

Talk is being recorded.

Covered topics:

The category F

Let F be the category whose objects are the finite sets $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$ and morphisms al the maps $f : [n] \to [m]$.

The category F

Let F be the category whose objects are the finite sets $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$ and morphisms al the maps $f\colon [n] \to [m]$.

Definition

A representation of F is a functor $V\colon \mathrm{F} \to \mathrm{Vec}_{\mathbb{Q}}$, the category of (finite dimensional) $\mathbb{Q}$-vector spaces.

Equivalently, a representation is a collection of (finite dimensional) vector spaces $(V[n])_{n\in\mathbb{N}}$ and linear maps $f_* : V[n] \to V[m]$ for each map $f \in \mathrm{Map}([n],[m])$ such that $(g \circ f)_* = g_* \circ f_*$.

The category F

Let F be the category whose objects are the finite sets $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$ and morphisms al the maps $f \colon [n] \to [m]$.

Definition

A representation of F is a functor $V \colon \mathrm{F} \to \mathrm{Vec}_{\mathbb{Q}}$, the category of (finite dimensional) $\mathbb{Q}$-vector spaces.

Equivalently, a representation is a collection of (finite dimensional) vector spaces $(V[n])_{n\in\mathbb{N}}$ and linear maps $f_* : V[n] \to V[m]$ for each map $f \in \mathrm{Map}([n], [m])$ such that $(g \circ f)_* = g_* \circ f_*$.

Example

Let D_0 be the representation given by $D_0[0] = \mathbb{Q}$ and $D_0[n] = 0$ for $n > 0$. This is a representation because $\mathrm{Map}([n], [0]) = \emptyset$ for $n > 0$.

Classical representation theory

For each representation V and $n \in \mathbb{N}$, the vector space $V[n]$ is a representation of $\mathfrak{S}_n$, i.e. the group of bijections $[n] \to [n]$.

Classical representation theory

For each representation V and $n \in \mathbb{N}$, the vector space $V[n]$ is a representation of $\mathfrak{S}_n$, i.e. the group of bijections $[n] \to [n]$.

Theorem (Schur-Weyl duality)

Let W be a finite dimensional vector space, then there exists an isomorphism of $\mathrm{GL}(W) \times \mathfrak{S}_n$-representations:

$$W^{\otimes n} = \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda) \leq \dim W}} \mathbb{S}^\lambda(W) \boxtimes V_\lambda$$

where V_λ is an irreducible representation of $\mathfrak{S}_n$ indexed by a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$ of n (i.e. $\lambda_1 + \cdots + \lambda_{l(\lambda)} = n$) and $\mathbb{S}^\lambda$ is the Schur functor.

Classical representation theory

For each representation V and $n \in \mathbb{N}$, the vector space $V[n]$ is a representation of $\mathfrak{S}_n$, i.e. the group of bijections $[n] \to [n]$.

Theorem (Schur-Weyl duality)

Let W be a finite dimensional vector space, then there exists an isomorphism of $\mathrm{GL}(W) \times \mathfrak{S}_n$*-representations:*

$$W^{\otimes n} = \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda) \leq \dim W}} \mathbb{S}^\lambda(W) \boxtimes V_\lambda$$

where V_λ is an irreducible representation of $\mathfrak{S}_n$ indexed by a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{l(\lambda)})$ of n (i.e. $\lambda_1 + \cdots + \lambda_{l(\lambda)} = n$) and $\mathbb{S}^\lambda$ is the Schur functor.

Example

For $n = 2$ we have $W \otimes W = S^2 W \oplus \Lambda^2 W$.

The indecomposable projective representations

Definition

A *Schur projective representation* of weight k is $\mathbb{P}_\lambda$ for $\lambda \vdash k$ defined by

$$\mathbb{P}_\lambda[n] = \mathbb{S}^\lambda(\mathbb{Q}^n)$$

and each $f\colon [n] \to [m]$ induces $\tilde{f}\colon \mathbb{Q}^n \to \mathbb{Q}^m$ and the linear map $\mathbb{S}^\lambda(\tilde{f})\colon \mathbb{P}_\lambda[n] \to \mathbb{P}_\lambda[m]$.

The dimension of $\mathbb{P}_\lambda[n]$ is given by the Schur polynomial $s_\lambda(1^n)$.

The indecomposable projective representations

Definition

A *Schur projective representation* of weight k is $\mathbb{P}_\lambda$ for $\lambda \vdash k$ defined by

$$\mathbb{P}_\lambda[n] = \mathbb{S}^\lambda(\mathbb{Q}^n)$$

and each $f\colon [n] \to [m]$ induces $\tilde{f}\colon \mathbb{Q}^n \to \mathbb{Q}^m$ and the linear map $\mathbb{S}^\lambda(\tilde{f})\colon \mathbb{P}_\lambda[n] \to \mathbb{P}_\lambda[m]$.

The dimension of $\mathbb{P}_\lambda[n]$ is given by the Schur polynomial $s_\lambda(1^n)$.

Example

The exact sequence of representations

$$\cdots \to \mathbb{P}_{1^3} \to \mathbb{P}_{1^2} \to \mathbb{P}_1 \to \mathbb{P}_0 \to 0$$

specialize on the object $[n]$ to the Koszul complex

$$\cdots \to \Lambda^3\mathbb{Q}^n \to \Lambda^2\mathbb{Q}^n \to \mathbb{Q}^n \to \mathbb{Q} \to 0.$$

The representations D_k

Definition

Let D_k be the kernel $\ker(\mathbb{P}_{1^{k-1}} \to \mathbb{P}_{1^{k-2}})$

The dimension of $D_k[n] = V_{(n-k+1,1^{k-1})}$ is $\binom{n-1}{k-1}$ (for $k > 0$).

The representations D_k

Definition

Let D_k be the kernel $\ker(\mathbb{P}_{1^{k-1}} \to \mathbb{P}_{1^{k-2}})$

The dimension of $D_k[n] = V_{(n-k+1,1^{k-1})}$ is $\binom{n-1}{k-1}$ (for $k > 0$).

Example

The representation $\mathbb{P}_1$ has D_2 as a subrepresentation:

$$0 \to D_2 \to \mathbb{P}_1 \to D_1 \to 0.$$

On the object $[n]$ (for $n > 0$) is

$$0 \to V \to \mathbb{Q}^n \to \mathbb{Q} \to 0$$

where $V = \langle e_i - e_j \rangle$. The sequence of F-representation do not split (using $i_1, i_2 \colon [1] \to [2]$ the retraction $r \colon \mathbb{Q} \to \mathbb{Q}^2$ would satisfy $e_1 = i_1(r(1)) = r(1) = i_2(r(1)) = e_2$).

The representations C_λ

Definition

Let $\lambda \vdash k$ with $\lambda_1 > 1$, the representation C_λ is defined by $C_\lambda[n] = 0$ for $n < k$ and $C_\lambda[n] = \mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n} V_\lambda \boxtimes 1_{n-k}$.

The representations C_λ

Definition

Let $\lambda \vdash k$ with $\lambda_1 > 1$, the representation C_λ is defined by $C_\lambda[n] = 0$ for $n < k$ and $C_\lambda[n] = \operatorname{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n} V_\lambda \boxtimes 1_{n-k}$.

The dimension of $C_\lambda[n]$ is $\binom{n}{k} \dim V_\lambda = \binom{n}{k} \langle s_\lambda, p_{1^k} \rangle$.
We say that C_λ is of weight $k = |\lambda|$.

Some facts

Definition

A representation V is *finitely generated* if there is a finite set $\{v_i\}_{i=1,\dots,N}$ of elements $v_i \in V[n_i]$ such that $\langle v_i \rangle_{i=1,\dots,N} = V$.

Some facts

Definition

A representation V is *finitely generated* if there is a finite set $\{v_i\}_{i=1,\dots,N}$ of elements $v_i \in V[n_i]$ such that $\langle v_i\rangle_{i=1,\dots,N} = V$.

Theorem (Wiltshire-Gordon '14)

- $\mathrm{Hom}_{\mathrm{F}}(\mathbb{P}_\lambda, V) \cong \mathrm{Hom}_{\mathfrak{S}_k}(V_\lambda, V[k])$.
- *The category of f.g. representations has the Jordan-Hölder property.*
- *The indecomposable projective are* $\{\mathbb{P}_\lambda\}_\lambda$.
- *The irreducible representations are* $\{D_k\}_{k\in\mathbb{N}}$ *and* $\{C_\lambda\}_{\lambda_1>1}$.
- *Let* V *be f.g., the sequence* $\dim V[n]$ *is polynomial in* n *for* $n > 0$.

Definition

The *skeleton filtration* of V is the filtration $\{\mathrm{sk}_k V\}_k$ defined by $\mathrm{sk}_k V = \langle V[i] \rangle_{i \leq k}$.

Definition

The *skeleton filtration* of V is the filtration $\{\mathrm{sk}_k V\}_k$ defined by $\mathrm{sk}_k V = \langle V[i] \rangle_{i \leq k}$.

Example

Recall that $\mathbb{P}_1[n] = \mathbb{Q}^n$, we have $\mathrm{sk}_k \mathbb{P}_1 = \mathbb{P}_1$ for $k \geq 1$ and $\mathrm{sk}_k \mathbb{P}_1 = 0$ otherwise. Therefore, $\mathrm{gr}^1_{\mathrm{sk}} \mathbb{P}_1 = \mathbb{P}_1$ is not semisimple. Moreover the inclusion $i\colon D_2 \hookrightarrow \mathbb{P}_1$ induces the zero map $\mathrm{gr}_{\mathrm{sk}}(i) = 0$ (indeed $\mathrm{gr}^2_{\mathrm{sk}} D_2 = D_2$).

Definition

The *skeleton filtration* of V is the filtration $\{\mathrm{sk}_k V\}_k$ defined by $\mathrm{sk}_k V = \langle V[i] \rangle_{i \leq k}$.

Example

Recall that $\mathbb{P}_1[n] = \mathbb{Q}^n$, we have $\mathrm{sk}_k \mathbb{P}_1 = \mathbb{P}_1$ for $k \geq 1$ and $\mathrm{sk}_k \mathbb{P}_1 = 0$ otherwise. Therefore, $\mathrm{gr}^1_{\mathrm{sk}} \mathbb{P}_1 = \mathbb{P}_1$ is not semisimple. Moreover the inclusion $i \colon D_2 \hookrightarrow \mathbb{P}_1$ induces the zero map $\mathrm{gr}_{\mathrm{sk}}(i) = 0$ (indeed $\mathrm{gr}^2_{\mathrm{sk}} D_2 = D_2$).

Lemma (P. '20)

Let $f \colon V \to W$ be a morphism of F-representations. Suppose that V does not have composition factors of type D. Then

- $\mathrm{gr}_{\mathrm{sk}} V$ *is semisimple,*
- *the map f and $\mathrm{gr}_{\mathrm{sk}} f$ have the same rank.*

Computing resolutions

Theorem (Assaf, Speyer '18, Ryba '18)

The minimal projective resolution of C_λ, $\lambda \vdash k$, is given by

$$0 \to \mathbb{P}^1 \to \cdots \to \mathbb{P}^{k-1} \to \mathbb{P}_\lambda \to C_\lambda \to 0,$$

where $\mathbb{P}^n = \bigoplus_{\mu \vdash n} \mathbb{P}_\mu^{c_\lambda^\mu}$ *and the coefficients are*

$$c_\lambda^\mu = \langle s_{\lambda'}, s_{\mu'}[L] \rangle,$$

where L is the Lyndon symmetric function (i.e., the character of the free Lie algebra).

Computing resolutions

Theorem (Assaf, Speyer '18, Ryba '18)

The minimal projective resolution of C_λ, $\lambda \vdash k$, is given by

$$0 \to \mathbb{P}^1 \to \cdots \to \mathbb{P}^{k-1} \to \mathbb{P}_\lambda \to C_\lambda \to 0,$$

where $\mathbb{P}^n = \bigoplus_{\mu \vdash n} \mathbb{P}_\mu^{c_\lambda^\mu}$ and the coefficients are

$$c_\lambda^\mu = \langle s_{\lambda'}, s_{\mu'}[L] \rangle,$$

where L is the Lyndon symmetric function (i.e., the character of the free Lie algebra).

Example

We have

$$0 \to \mathbb{P}_{(1,1)} \to \mathbb{P}_{(2,1)} \oplus \mathbb{P}_{(3)} \to \mathbb{P}_{(4)} \to C_{(4)} \to 0,$$

and so $\mathrm{Ext}^1(C_{(4)}, C_{(2)}) = \mathbb{Q}$.

Ordered configuration spaces

Let X be a topological space. Define:

$$\mathrm{Conf}_n(X) := \{(p_1, \ldots, p_n) \in X^n \mid p_i \neq p_j\}$$

Ordered configuration spaces

Let X be a topological space. Define:

$$\mathrm{Conf}_n(X) := \{(p_1, \ldots, p_n) \in X^n \mid p_i \neq p_j\}$$

Example

$\mathrm{Conf}_n(S^1) = S^1 \times \mathfrak{S}_{n-1} \times \mathbb{R}^{n-1}$.

Ordered configuration spaces

Let X be a topological space. Define:

$$\mathrm{Conf}_n(X) := \{(p_1, \ldots, p_n) \in X^n \mid p_i \neq p_j\}$$

Example

$\mathrm{Conf}_n(S^1) = S^1 \times \mathfrak{S}_{n-1} \times \mathbb{R}^{n-1}$.

Example

$\mathrm{Conf}_n(\mathbb{R}^2)$ is the complement of the hyperplane arrangement of type A_{n-1}.

Delete a point

Theorem (Fadell, Neuwirth 1962)

If M is a manifold without boundary, then $p\colon \mathrm{Conf}_n(M) \to \mathrm{Conf}_{n-1}(M)$ is a fibration with fibre $M \setminus \{n-1 \text{ points}\}$.

Delete a point

Theorem (Fadell, Neuwirth 1962)

If M is a manifold without boundary, then $p\colon \mathrm{Conf}_n(M) \to \mathrm{Conf}_{n-1}(M)$ is a fibration with fibre $M \setminus \{n-1 \text{ points}\}$.

Recall the long exact sequence of homotopy groups:

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \dots$$

Delete a point

Theorem (Fadell, Neuwirth 1962)

If M is a manifold without boundary, then $p\colon \mathrm{Conf}_n(M) \to \mathrm{Conf}_{n-1}(M)$ is a fibration with fibre $M \setminus \{n-1 \text{ points}\}$.

Recall the long exact sequence of homotopy groups:

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \dots$$

Corollary (Fadell, Neuwirth 1962)

If S is a surface different from S^2 and $\mathbb{P}_2(\mathbb{R})$, then $\mathrm{Conf}_n(S)$ is a $K(\pi, 1)$.

Add a point

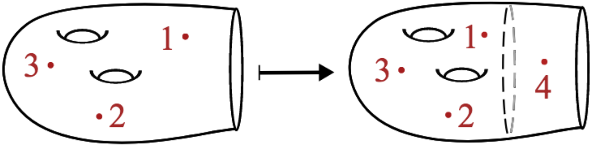
Theorem (Fadell, Neuwirth 1962)

If M is a non-compact manifold without boundary then the fibration $p\colon \mathrm{Conf}_n(M) \to \mathrm{Conf}_{n-1}(M)$ has a section.

Add a point

Theorem (Fadell, Neuwirth 1962)

If M is a non-compact manifold without boundary then the fibration $p\colon \mathrm{Conf}_n(M) \to \mathrm{Conf}_{n-1}(M)$ has a section.



The Euler characteristic

Theorem (Felix, Thomas 2000)

Let M be an even-dimensional manifold. Then

$$\sum_{n=0}^{\infty} \frac{\chi(\mathrm{Conf}_n(M))}{n!} u^n = (1+u)^{\chi(M)}$$

Theorem (Ellenberg, Wiltshire-Gordon 2015)

If M is a manifold that admits a non-zero vector field (i.e. $\chi(M) = 0$) then $\dim H^i(\mathrm{Conf}_n(M); \mathbb{Q})$ *is polynomial in n, for $n > 0$.*

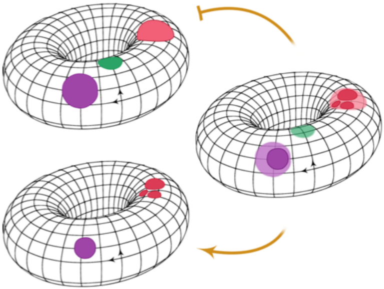
Theorem (Ellenberg, Wiltshire-Gordon 2015)

If M is a manifold that admits a non-zero vector field (i.e. $\chi(M) = 0$) then $\dim H^i(\mathrm{Conf}_n(M); \mathbb{Q})$ *is polynomial in n, for $n > 0$.*
Moreover, if M has two linearly independent vector fields (e.g. a complex variety with $\chi(M) = 0$) then $\bigoplus_n H^i(\mathrm{Conf}_n(M); \mathbb{Q})$ *is an* F*-representation.*

Theorem (Ellenberg, Wiltshire-Gordon 2015)

If M is a manifold that admits a non-zero vector field (i.e. $\chi(M) = 0$) then $\dim H^i(\mathrm{Conf}_n(M); \mathbb{Q})$ *is polynomial in n, for $n > 0$.*
Moreover, if M has two linearly independent vector fields (e.g. a complex variety with $\chi(M) = 0$) then $\bigoplus_n H^i(\mathrm{Conf}_n(M); \mathbb{Q})$ *is an* F*-representation.*

The map $f\colon [4] \to [3]$ is defined by $f(1) = 1$, $f(2) = f(3) = f(4) = 2$.



The Kriz model

Theorem (Kriz '94, Totaro '96)

Let M be a smooth projective variety. There exists a dga $(E(M), \mathrm{d})$ such that $H^\bullet(E_n(M), \mathrm{d}) \simeq H^\bullet(\mathrm{Conf}_n(M); \mathbb{Q})$.

The Kriz model

Theorem (Kriz '94, Totaro '96)

Let M be a smooth projective variety. There exists a dga $(E(M), \mathrm{d})$ such that $H^\bullet(E_n(M), \mathrm{d}) \simeq H^\bullet(\mathrm{Conf}_n(M); \mathbb{Q})$.

Let $E_n(M)$ be the exterior algebra on generators

- x_i for x in a basis of $H^\bullet(M)$ and $i \leq n$ with degree $(\deg x, 0)$,
- $G_{i,j}$ for $i < j$ with degree $(0, d-1)$,

The Kriz model

Theorem (Kriz '94, Totaro '96)

Let M be a smooth projective variety. There exists a dga $(E(M), \mathrm{d})$ such that $H^\bullet(E_n(M), \mathrm{d}) \simeq H^\bullet(\mathrm{Conf}_n(M); \mathbb{Q})$.

Let $E_n(M)$ be the exterior algebra on generators

- x_i for x in a basis of $H^\bullet(M)$ and $i \leq n$ with degree $(\deg x, 0)$,
- $G_{i,j}$ for $i < j$ with degree $(0, d-1)$,

and relations

- $(x_i - x_j)G_{i,j} = 0$,
- $G_{i,j}G_{j,k} - G_{i,j}G_{i,k} + G_{j,k}G_{i,k} = 0$.

The Kriz model

Theorem (Kriz '94, Totaro '96)

Let M be a smooth projective variety. There exists a dga $(E(M), \mathrm{d})$ such that $H^\bullet(E_n(M), \mathrm{d}) \simeq H^\bullet(\mathrm{Conf}_n(M); \mathbb{Q})$.

Let $E_n(M)$ be the exterior algebra on generators

- x_i for x in a basis of $H^\bullet(M)$ and $i \leq n$ with degree $(\deg x, 0)$,
- $G_{i,j}$ for $i < j$ with degree $(0, d-1)$,

and relations

- $(x_i - x_j)G_{i,j} = 0$,
- $G_{i,j}G_{j,k} - G_{i,j}G_{i,k} + G_{j,k}G_{i,k} = 0$.

The differential of degree $(d, 1-d)$ is given by

- $\mathrm{d}(x_i) = 0$,
- $\mathrm{d}(G_{i,j}) = [\Delta]_{i,j}$.

The action of F

If $\chi(M) = 0$, the Kriz model $E(M) = \bigoplus_{n \in \mathbb{N}} E_n(M)$ is a representation of F with the action of $f : [n] \to [m]$ given by:

$$f_*(x_i) = x_{f(i)} \qquad f_*(G_{i,j}) = \begin{cases} 0 & \text{if } f(i) = f(j) \\ G_{f(i),f(j)} & \text{otherwise} \end{cases}$$

The action of F

If $\chi(M) = 0$, the Kriz model $E(M) = \bigoplus_{n\in\mathbb{N}} E_n(M)$ is a representation of F with the action of $f\colon [n] \to [m]$ given by:

$$f_*(x_i) = x_{f(i)} \qquad f_*(G_{i,j}) = \begin{cases} 0 & \text{if } f(i) = f(j) \\ G_{f(i),f(j)} & \text{otherwise} \end{cases}$$

Indeed, if $f(i) = f(j)$ we have $\mathrm{d}(f_*(G_{i,j})) = 0$ and

$$f_*(\mathrm{d}(G_{i,j})) = f_*([\Delta]_{i,j}) = \chi(M)[M]_{f(i)}.$$

Representation theory of the Kriz model

Let $\mathcal{E} = S^1 \times S^1$ be an elliptic curve.
Notice that $E_n^{0,q}(\mathcal{E})$ is the Orlik-Solomon algebra for the arrangement of type A_n. So

$$E_n^{0,n-1}(\mathcal{E}) = \operatorname{Ind}_{C_n}^{\mathfrak{S}_n} \zeta_n$$

as representation of $\mathfrak{S}_n$ with basis the descending trees on $[n]$.

Representation theory of the Kriz model

Let $\mathcal{E} = S^1 \times S^1$ be an elliptic curve.
Notice that $E_n^{0,q}(\mathcal{E})$ is the Orlik-Solomon algebra for the arrangement of type A_n. So

$$E_n^{0,n-1}(\mathcal{E}) = \operatorname{Ind}_{C_n}^{\mathfrak{S}_n} \zeta_n$$

as representation of $\mathfrak{S}_n$ with basis the descending trees on $[n]$.

Theorem (Stanley '82, Lehrer, Solomon '86, Ashraf, Azam, Berceanu '12)

The action of $\mathfrak{S}_n$ on the Kriz model is

$$E_n^{p,q}(\mathcal{E}) \cong \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda)=n-q \\ w(\lambda)=p}} \operatorname{Ind}_{Z(\lambda)}^{\mathfrak{S}_n} \zeta_\lambda$$

where the sum is taken over all labelled partitions λ with blocks label by $\{1, x, y, xy\}$. The number $w(\lambda)$ is the sum of the degree of the labels.

Lemma (P. '20)

The sign representation appears only in the first row $E_n^{\bullet,0}(M)$ *(M even dimensional).*

Lemma (P. '20)

The sign representation appears only in the first row $E_n^{\bullet,0}(M)$ (M even dimensional).

Theorem (P. '20)

For $q > 0$ the composition factors are

$$E^{p,q}(\mathcal{E}) \sim \bigoplus_{\substack{|\lambda|-l(\lambda)=q \\ w(\lambda)=p}} C_{\mathrm{Ind}_{Z(\lambda)}^{\mathfrak{S}_n} \zeta_\lambda},$$

where the sum is taken over all partitions λ without blocks of size one and label 1. Moreover

$$\mathrm{gr}_{\mathrm{sk}}^n E^{p,q}(\mathcal{E}) = \bigoplus_{\substack{|\lambda|=n \\ |\lambda|-l(\lambda)=q \\ w(\lambda)=p}} C_{\mathrm{Ind}_{Z(\lambda)}^{\mathfrak{S}_n} \zeta_\lambda}.$$

We have

$$\operatorname{gr}^{\bullet}_{\mathsf{sk}} H(E^{\bullet,\bullet}(\mathcal{E}), \mathrm{d}) \cong H(\operatorname{gr}^{\bullet}_{\mathsf{sk}} E^{\bullet,\bullet}(\mathcal{E}), \operatorname{gr} \mathrm{d})$$

We have

$$\operatorname{gr}^\bullet_{\mathrm{sk}} H(E^{\bullet,\bullet}(\mathcal{E}), \mathrm{d}) \cong H(\operatorname{gr}^\bullet_{\mathrm{sk}} E^{\bullet,\bullet}(\mathcal{E}), \operatorname{gr}\mathrm{d})$$

Example

The differential is $\mathrm{d}(G_{i,j}) = x_i y_i - x_i y_j - x_j y_i + x_j y_j$ and the graded differential is $\operatorname{gr}\mathrm{d}(G_{i,j}) = -x_i y_j - x_j y_i$

We have

$$\mathrm{gr}^{\bullet}_{\mathrm{sk}} H(E^{\bullet,\bullet}(\mathcal{E}), \mathrm{d}) \cong H(\mathrm{gr}^{\bullet}_{\mathrm{sk}} E^{\bullet,\bullet}(\mathcal{E}), \mathrm{gr}\,\mathrm{d})$$

Example

The differential is $\mathrm{d}(G_{i,j}) = x_i y_i - x_i y_j - x_j y_i + x_j y_j$ and the graded differential is $\mathrm{gr}\,\mathrm{d}(G_{i,j}) = -x_i y_j - x_j y_i$

We have $\mathrm{gr}^n E^{p,q}(\mathcal{E}) = 0$ for $n \leq q$ or $2(n-q) < p$ or $n > 2q + p$.

We have

$$\mathrm{gr}^{\bullet}_{\mathrm{sk}} H(E^{\bullet,\bullet}(\mathcal{E}), \mathrm{d}) \cong H(\mathrm{gr}^{\bullet}_{\mathrm{sk}} E^{\bullet,\bullet}(\mathcal{E}), \mathrm{gr}\,\mathrm{d})$$

Example

The differential is $\mathrm{d}(G_{i,j}) = x_i y_i - x_i y_j - x_j y_i + x_j y_j$ and the graded differential is $\mathrm{gr}\,\mathrm{d}(G_{i,j}) = -x_i y_j - x_j y_i$

We have $\mathrm{gr}^n E^{p,q}(\mathcal{E}) = 0$ for $n \leq q$ or $2(n-q) < p$ or $n > 2q + p$.

Corollary (P. '20)

For $q > 0$ we have as $\mathfrak{S}_n$-representation

$$H^{p,q}(E_n^{\bullet,\bullet}(\mathcal{E}), \mathrm{d}) = \bigoplus_k \mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n} H^{p,q}(\mathrm{gr}^k_{\mathrm{sk}} E_k^{\bullet,\bullet}(\mathcal{E}), \mathrm{gr}\,\mathrm{d}) \boxtimes 1_{n-k}.$$

For $q = 0$ we have

$$H^{p,0}(E_n^{\bullet,\bullet}(\mathcal{E}), \mathrm{d}) = \mathbb{P}_{1^p} \boxtimes \mathbb{V}_p \oplus \mathbb{P}_{1^{p-1}} \boxtimes \mathbb{V}_{p-2}$$

Lemma

Let $f : \mathbb{N}_0 \to \mathbb{N}$ *be a polynomial function, i.e.* $f \in \mathbb{Q}[x]$*. Then* $f(n) = \sum_{i=0}^{\deg f} a_i \binom{n}{i}$ *for some unique integer coefficients* $a_i \in \mathbb{Z}$*.*

Lemma

Let $f : \mathbb{N}_0 \to \mathbb{N}$ be a polynomial function, i.e. $f \in \mathbb{Q}[x]$. Then $f(n) = \sum_{i=0}^{\deg f} a_i \binom{n}{i}$ for some unique integer coefficients $a_i \in \mathbb{Z}$.

Example

For $M = S^3$ we have $H^2(\mathrm{Conf}_n(S^3)) = H^{0,2}(E_n^{\bullet,\bullet}(S^3), \mathrm{d})$ and $H^2(\mathrm{Conf}_\bullet(S^3)) = D_3$ as F-module. Therefore $\dim H^2(\mathrm{Conf}_n(S^3)) = \binom{n}{2} - \binom{n}{1} + \binom{n}{0}$.

Lemma

Let $f : \mathbb{N}_0 \to \mathbb{N}$ be a polynomial function, i.e. $f \in \mathbb{Q}[x]$. Then $f(n) = \sum_{i=0}^{\deg f} a_i \binom{n}{i}$ for some unique integer coefficients $a_i \in \mathbb{Z}$.

Example

For $M = S^3$ we have $H^2(\mathrm{Conf}_n(S^3)) = H^{0,2}(E_n^{\bullet,\bullet}(S^3), \mathrm{d})$ and $H^2(\mathrm{Conf}_\bullet(S^3)) = D_3$ as F-module. Therefore $\dim H^2(\mathrm{Conf}_n(S^3)) = \binom{n}{2} - \binom{n}{1} + \binom{n}{0}$.

Corollary (P. '20)

For all n we have

$$\dim H^{p,q}(E_n^{\bullet,\bullet}(\mathcal{E}), \mathrm{d}) = \sum_k \binom{n}{k} \dim H^{p,q}(\mathrm{gr}_{\mathrm{sk}}^k E^{\bullet,\bullet}(\mathcal{E}), \mathrm{gr}\,\mathrm{d}).$$

Theorem (P. '20)

The Betti numbers of $\mathrm{Conf}_n(\mathcal{E})$ *are:*

$$
\begin{aligned}
b_0 &= 1,\\
b_1 &= 2n,\\
b_2 &= 2\binom{n}{3} + 3\binom{n}{2} + n,\\
b_3 &= 14\binom{n}{4} + 8\binom{n}{3} + 2\binom{n}{2},\\
b_4 &= 32\binom{n}{6} + 74\binom{n}{5} + 32\binom{n}{4} + 5\binom{n}{3},\\
b_5 &= 63\binom{n}{8} + 427\binom{n}{7} + 490\binom{n}{6} + 154\binom{n}{5} + 18\binom{n}{4},\\
b_k &= c_k\binom{n}{2k-2} + o(n^{2k-2}),
\end{aligned}
$$

where $c_k \geq \binom{2k-3}{k-3}$.

Conjecture

The coefficient c_k is equal to $\binom{2k-3}{k-3}$.

Conjecture

The coefficient c_k is equal to $\binom{2k-3}{k-3}$.

From the previous theorem, for $k > 5$ we have $\mathrm{gr}_{\mathrm{sk}}^{2k} H^k(E^{\bullet,\bullet}(\mathcal{E})) = 0$ and $\mathrm{gr}_{\mathrm{sk}}^{2k-1} H^k(E^{\bullet,\bullet}(\mathcal{E})) = 0$. Moreover

$$\mathrm{gr}_{\mathrm{sk}}^{2k-2} H^k(E^{\bullet,\bullet}(\mathcal{E})) = \mathrm{gr}_{\mathrm{sk}}^{2k-2} H^{2,k-2}(E^{\bullet,\bullet}(\mathcal{E}))$$

that contains $C_{(k+1,1^{k-3})}$ ($c_k = \dim V_{(k+1,1^{k-3})}$).

Thanks for listening!

roberto.pagaria@unibo.it