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Combinatorial decomposition theorem for Hitchin systems via zonotopes

Joint with M. Mauri and L. Migliorini

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Covered topics:

Consider a undirected graph Γ on the vertex set $[r] = \{1, \dots, r\}$ with y_{ij} edges between vertices i and j.

Definition

The *graphical zonotope* Z_Γ of Γ is the integral polytope defined by the Minkowski sum:

$$Z_\Gamma := \sum_{(i,j)\in\Gamma} y_{ij}[0, e_i - e_j] \subset \mathbb{R}^r.$$

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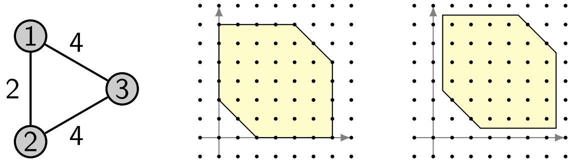
For any polytope Z let $C(Z)$ be the number of integer points in the interior of Z.

Example



we have $C(Z_\Gamma) = 23$ and $C(Z_\Gamma + (1/2, 1/2)) = 30$.

We consider graphs Γ possibly with multiple edges. A *flat* is a partition of $[r]$ such that for each block the induced subgraph is connected. The *poset of flats* $\mathcal{S}$ is the set of all flats ordered by refinement.

Definition

Let $\underline{S} \in \mathcal{S}$ be a flat, the *deleted* graph $\Gamma_{\underline{S}}$ is the graph with only edges in the flat $\underline{S}$. The *contracted* graph $\Gamma^{\underline{S}}$ is obtained from Γ by contracting all the edges in the flat $\underline{S}$.

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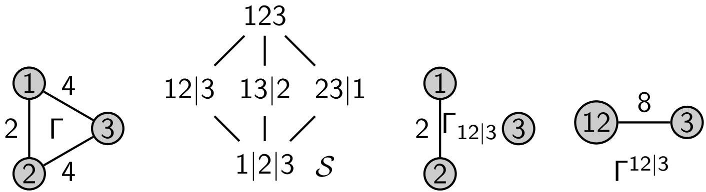
Example

Consider the graph Γ with poset of flats $\mathcal{S}$ and the flat $12|3$.

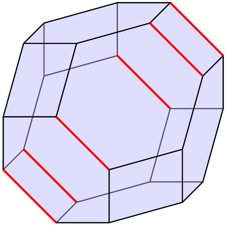


Faces of zonotopes

Proposition

Every face of Z_Γ is a (translated) graphical zonotope $Z_{\Gamma_{\underline{S}}}$ for some flat $\underline{S} \in \mathcal{S}$. Moreover, the number of faces parallel to $Z_{\Gamma_{\underline{S}}}$ is equal to the number of acyclic orientations of $\Gamma^{\underline{S}}$.

The vertices of Z_Γ are in bijection with acyclic orientations of Γ.
In the
example $\Gamma = K_4$ and Z_{K_4} is the *permutohedron*.
In red the faces parallel to the segment $Z_{\Gamma_{12|3|4}}$.
The number of such faces is equal to the
number of acyclic orientations of $\Gamma^{12|3|4} \sim K_3$.



Goal

Consider a graph Γ on vertices $[r]$ and y_{ij} edges between i and j. Let $\omega \in \mathbb{R}^r$ be a vector. We want to express $C(Z_\Gamma + \omega)$ in term of the numbers $C(Z_{\Gamma_{\underline{S}}})$ for all $\underline{S} \in \mathcal{S}$.
More precisely,

$$C(Z_\Gamma + \omega) = \sum_{\underline{S} \in \mathcal{S}} c_{\underline{S},\omega} C(Z_{\Gamma_{\underline{S}}})$$

where the coefficients $c_{\underline{S},\omega}$ do not depend on y_{ij} but only on the poset of flats $\mathcal{S}$ (i.e. on $\delta_{y_{ij}=0}$).

Counting integer points

Theorem (Stanley '91, Ardila Beck McWhirter '20)

Let $Z = \sum_i [0, v_i]$ be an integral zonotope and $\omega \in \mathbb{R}^r$. Then

$$C(Z+\omega) = \sum_{I \text{ independent set}} (-1)^{r-|I|} \delta_{(\langle v_i \rangle_{i\in I} + \omega)\cap \mathbb{Z}^r \neq \emptyset} \mathrm{Vol}(I).$$

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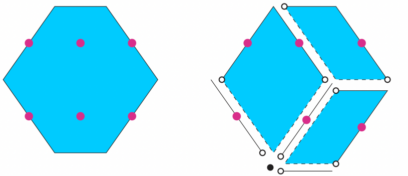
Example

Let $Z = [0, e_1] + [0, e_1 + e_2] + [0, e_1 - e_2]$ and $\omega = (\frac{1}{2}, \frac{1}{2})$.



$$C(Z+\omega) = \operatorname{Vol}(v_2 v_3) + \operatorname{Vol}(v_1 v_2) + \operatorname{Vol}(v_1 v_3) - \operatorname{Vol}(v_2) - \operatorname{Vol}(v_3)$$
$$= 2 + 1 + 1 - 1 - 1 = 2.$$

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Definition

A set $S \subseteq [r]$ is ω-integral if $\sum_{i \in S} \omega_i \in \mathbb{Z}$. A partition $\underline{S} \vdash [r]$ is ω-integral if all its blocks S_j are ω-integral.

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For a graphical zonotope Z_Γ and a flat $\underline{S} \in \mathcal{S}$ we have $\delta_{(\langle \underline{S} \rangle + \omega) \cap \mathbb{Z}^r \neq \emptyset} = 1$ if and only if $\underline{S}$ is ω-integral.

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Möbius inversion

Theorem (Mauri, Migliorini, P. '23)

If $\sum_{i=1}^{r} \omega_i \in \mathbb{Z}$, *then*

$$C(Z_\Gamma + \omega) = C(Z_\Gamma) + \sum_{\underline{S} \in \mathcal{S}} \Big(\sum_{\substack{\underline{T} \geq \underline{S} \\ \underline{T}\ \omega\text{-integral}}} \mu_{\mathcal{S}}(\underline{S}, \underline{T}) \Big) C(Z_{\Gamma_{\underline{S}}}).$$

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Corollary

In the case of the complete graph Γ_r *we have*

$$c_{\hat{0},\omega} = \sum_{\substack{\underline{S} \vdash [r] \\ \underline{S}\ \omega\text{-integral}}} (-1)^{\ell(\underline{S})-1} \prod_{i=1}^{\ell(\underline{S})} (|S_i| - 1)!$$

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Moreover, $c_{\hat{0},\omega} = 0$ *if* $\omega \in \mathbb{Z}^r$.

Question: are the coefficients $c_{\underline{S},\omega}$ non-negative?

Shellability

We denote by $\mathcal{S}_\omega \subset \mathcal{S}$ the downward closed subposet of non-ω-integral flats. Let $\Delta(\mathcal{S}_\omega)$ be the the *order complex* of the poset $\mathcal{S}_\omega$.

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Theorem (Mauri, Migliorini, P. '23)

The poset $\mathcal{S}_\omega$ is LEX-shellable. Therefore,

$$C(Z_\Gamma + \omega) = C(Z_\Gamma) + \sum_{\underline{S} \in \mathcal{S}_\omega} \operatorname{rk} \widetilde{H}^{\mathrm{top}}(\Delta(\mathcal{S}_{\omega, \geq \underline{S}}))C(Z_{\Gamma_{\underline{S}}}).$$

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Corollary

If $\omega \notin \mathbb{Z}^r$, then $c_{\hat{0},\omega} \neq 0$.

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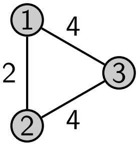
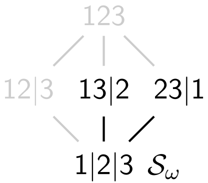
Example

Let $\omega = (\frac{1}{2}, \frac{1}{2}, 1)$ and Γ be the graph





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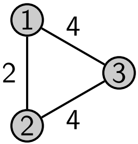

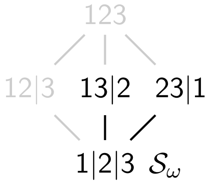

$$C(Z_\Gamma+\omega) = C(Z_\Gamma) + C(Z_{\Gamma_{13|2}}) + C(Z_{\Gamma_{23|1}}) + C(Z_{\Gamma_{1|2|3}})$$

$$30 = 23 + 3 + 3 + 1.$$

Orientation character

Let $O\Gamma$ be the oriented graph obtained by replacing every unoriented edge in Γ with the two possible oriented edges.

Definition

Consider the representation a_Γ of $\mathrm{Aut}(\Gamma)$ defined by

$$a_\Gamma(\sigma) = \mathrm{sgn}(\sigma\colon V(\Gamma) \to V(\Gamma))\,\mathrm{sgn}(\sigma\colon E(O\Gamma) \to E(O\Gamma))$$

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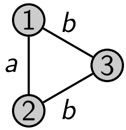

with $a \neq b$. Then $\mathrm{Aut}(\Gamma) = \mathbb{Z}/2\mathbb{Z} = \langle (12) \rangle$ and $a_\Gamma((12)) = (-1)^{a+1}$.

Permutation representations

Consider the group $\mathrm{Aut}(\Gamma) < \mathfrak{S}_r$ and suppose that ω is a $\mathrm{Aut}(\Gamma)$-invariant vector. Let $\mathcal{C}(Z_\Gamma + \omega)$ be the permutation representation of $\mathrm{Aut}(\Gamma)$ on the set of integral points in the interior of $Z_\Gamma + \omega$

$$\dim \mathcal{C}(Z_\Gamma + \omega) = C(Z_\Gamma + \omega).$$

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Theorem (Mauri, Migliorini, P. 2023)

$$\mathcal{C}(Z_\Gamma + \omega) = \mathcal{C}(Z_\Gamma) \oplus \bigoplus_{\underline{S} \in \mathcal{S}_\omega / \mathrm{Aut}(\Gamma)} \mathrm{Ind}_{\mathrm{Stab}(\underline{S})}^{\mathrm{Aut}(\Gamma)} a_{\Gamma_{\underline{S}}} \otimes \widetilde{H}^{\mathrm{top}}(\Delta(\mathcal{S}_{\omega, \geq \underline{S}})) \otimes \mathcal{C}(\Gamma_{\underline{S}}).$$

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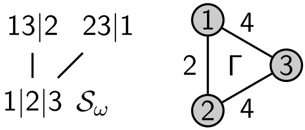
Example

Then $\omega = (\frac{1}{2}, \frac{1}{2}, 1)$ and



The automorphism group is $\operatorname{Aut}(\Gamma) = \mathbb{Z}/2\mathbb{Z} = \langle (12) \rangle$. Then:

$$\mathcal{C}(Z_\Gamma + \omega) = \mathcal{C}(Z_\Gamma) \oplus \operatorname{Reg}^{\oplus 3} \oplus (\operatorname{sgn} \otimes \operatorname{sgn} \otimes 1).$$

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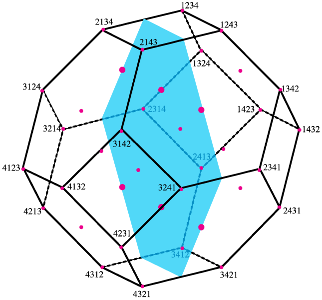

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Hitchin fibration

Let C be a smooth projective algebraic curve over $\mathbb{C}$ of genus $g_C > 1$ and $\mathcal{E}$ a vector bundle of rank n and degree d on C.

Definition

An *Higgs bundle* over C is a pair $(\mathcal{E}, \phi)$ where $\mathcal{E}$ is a vector bundle and $\phi \colon \mathcal{E} \to \mathcal{E} \otimes \omega_C$ an "endomorphism".
The *Dolbeault moduli space* is

$$M(n,d) = {\{\text{semistable Higgs bundle}\}}\big/{S\text{-equivalence}}.$$

Every endomorphisms has a characteristic polynomial.

Definition

The *Hitchin fibration* is the map

$$\chi \colon M(n,d) \to \mathbb{A}^N$$

sending $(\mathcal{E}, \phi)$ to the coefficients of char_ϕ.

Decomposition theorem

The space $M(n, d)$ is singular with a map to the affine space $\mathbb{A}^N$.
The cohomology does not work well on singular spaces, it is much better to consider the *intersection cohomology* $\mathrm{IH}(M(n, d))$.

$$\mathrm{IH}(M(n, d)) = H(M(n, d), \mathrm{IC}_{M(n,d)}) \simeq H(\mathbb{A}^N, R\chi_* \mathrm{IC}_{M(n,d)})$$

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Theorem (Mauri, Migliorini '22)

The Ngô Decomposition Theorem specializes to

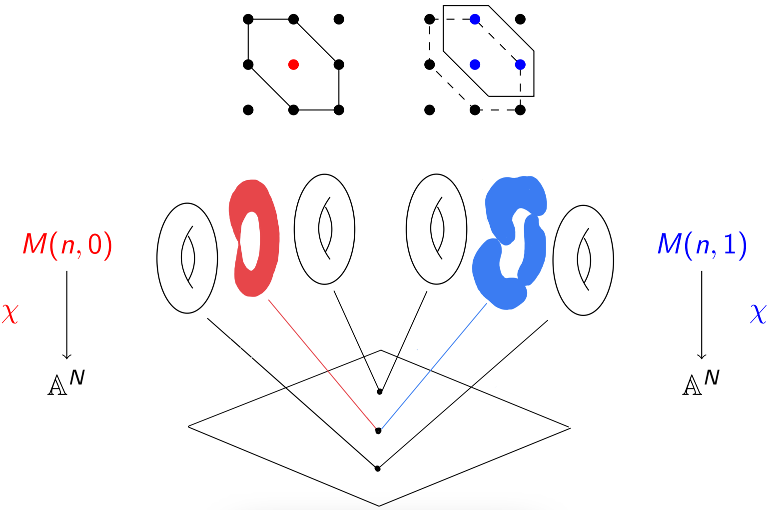
$$R\chi_* \mathrm{IC}_{M(n,d)}|_{\mathbb{A}^N_{\mathrm{red}}} = \bigoplus_{\underline{n} \vdash n} \mathrm{IC}_{S_{\underline{n}}}(\mathcal{L}_{\underline{n},d} \otimes \Lambda_{\underline{n}})$$

for some local systems $\mathcal{L}_{\underline{n},d}$ on $S_{\underline{n}}$ and for $\Lambda_{\underline{n}}$ the cohomology sheaf of the relative Picard group $\mathrm{Pic}^0(\overline{C}_{\underline{n}})$ of the normalization of the spectral curve.

Proposition

For any $a \in S_{\underline{n}}$ we have

$$\dim \mathcal{H}^{\mathrm{top}}(R\chi_* \mathrm{IC}_{M(n,d)})_a = \#\ \textit{irr. comp.}\ \chi^{-1}(a) = C(Z_{\Gamma_{\underline{n}}} + \omega)$$



$M(n,0)$
χ
$\mathbb{A}^N$
$M(n,1)$
χ
$\mathbb{A}^N$

Conclusions

Let $\underline{n} = \{n_1, n_2, \dots n_r\} \vdash n$ and $d \in \mathbb{N}$.
Problem: determine $\mathcal{L}_{\underline{n},d}$. In particular:

1. which partitions $\underline{n}$ appear in the decomposition (i.e. $\mathcal{L}_{\underline{n},d} \neq 0$)?
2. determine the rank $\mathrm{rk}(\mathcal{L}_{\underline{n},d})$.
3. determine the monodromy of the local system $\mathcal{L}_{\underline{n},d}$.

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Solution:

1. $\mathcal{L}_{\underline{n},d} \neq 0$ if and only if $\underline{n} = (n)$ or $\omega = (\frac{dn_i}{n}) \notin \mathbb{Z}^r$.
2. $$\mathrm{rk}(\mathcal{L}_{\underline{n},d}) = c_{\hat{0},\omega} = \sum_{\underline{S}\ \omega\text{-integral}} (-1)^{\ell(\underline{S})-1} \prod_i (|S_i| - 1)!$$
$$= \dim \widetilde{H}^{\mathrm{top}}(\Delta(\mathcal{S}_\omega)).$$
3. The monodromy is given by the representation of $\mathrm{Aut}(\Gamma_{\underline{n}})$
$$\mathrm{sgn} \otimes \widetilde{H}^{\mathrm{top}}(\Delta(\mathcal{S}_\omega)).$$

Thanks for listening!

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