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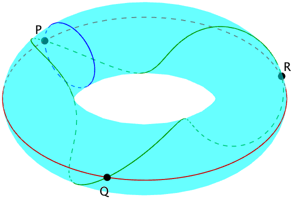
Counting Regions

PISA-HOKKAIDO-ROMA2 Summer School
on Mathematics and Its Applications 2018



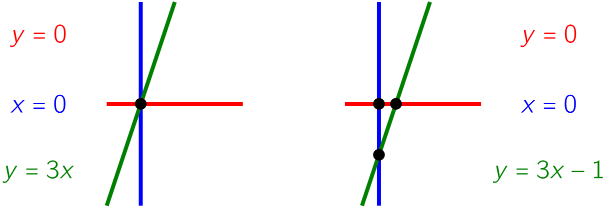
at Centro di Ricerca Matematica Ennio De Giorgi
September 6, 2018

Covered topics:

Consider an *arrangement* in $\mathbb{R}^n$, i.e. a finite set $\mathcal{A}$ of hyperplanes $H_1, \ldots, H_k$.

Example

Two arrangements of 3 lines in $\mathbb{R}^2$.



Problem: How many open *regions* (also called *chambers*) are described by the arrangement? How many of them are bounded?

Graded Poset

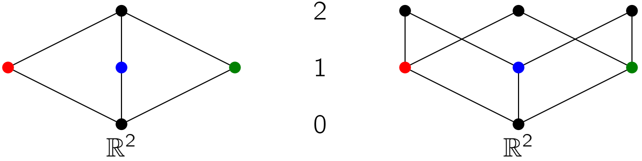
Definition

A *graded poset* is a partially ordered set $\mathcal{L}$ such that all maximal chains between any two elements have the same finite length. The *rank* of an element $x \in \mathcal{L}$ is the length of a maximal chain with x as a maximum.

The poset of *intersections* of an arrangement $\mathcal{A} = \{H_1, \ldots, H_k\}$ is the graded poset $\mathcal{L}(\mathcal{A})$ whose elements are the non-empty intersections of hyperplanes ordered by reverse inclusion.

Example

The posets of intersections of the previous example.



The Möbius Function

Let $\mathcal{L}$ be a (locally) finite poset.

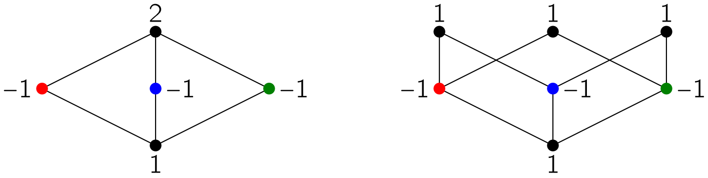
Definition

The *Möbius function* of $\mathcal{L}$ is $\mu_L : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}$ defined by:

$$\sum_{x \le z \le y} \mu_{\mathcal{L}}(x, z) = \delta_{x,y} \qquad \forall\, x, y \in \mathcal{L}.$$

Example

The Möbius functions $\mu(0, y)$, for $y \in \mathcal{L}$ of the two previous posets.



The classical Möbius function

Example

Let $(\mathbb{N}_{>0}, |)$ be the locally finite poset of positive integers ordered by the division relation. Then the function $\mu(1, n)$ is the classical Möbius function, i.e.:

$$\mu(1, n) = \begin{cases} 0 & \text{if } n \text{ is not squarefree} \\ 1 & \text{if } n \text{ has an even number of prime factors} \\ -1 & \text{if } n \text{ has an odd number of prime factors} \end{cases}$$

The Characteristic Polynomial

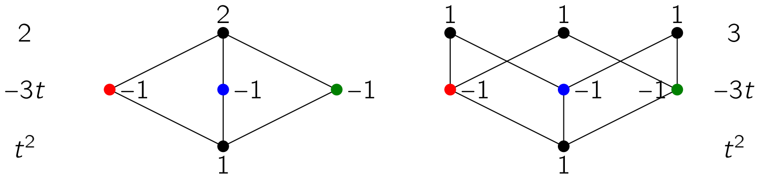
Definition

The *characteristic polynomial* of an hyperplane arrangement $\mathcal{A}$ is:

$$\chi_{\mathcal{A}}(t) \overset{\text{def}}{=} \sum_{x \in \mathcal{L}(\mathcal{A})} \mu(0,x) t^{n - \operatorname{rk} x}$$

Example

The characteristic polynomial $\chi(t)$ of the two examples are



respectively, $t^2 - 3t + 2$ and $t^2 - 3t + 3$.

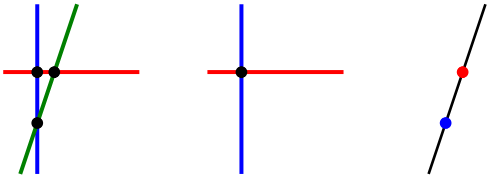
Deletion and Restriction

Definition

The *deletion* of H from $\mathcal{A}$ is the arrangement $\mathcal{A}_H = \mathcal{A} \setminus \{H\}$. The *restriction* of H is the arrangement $\mathcal{A}^H$ in the euclidean space H given by $\{K \cap H \mid K \in \mathcal{A}\}$.

Example

The arrangements $\mathcal{A}$, $\mathcal{A}_H$ and $\mathcal{A}^H$, respectively,



with polynomials $\chi_{\mathcal{A}} = t^2 - 3t + 3$, $\chi_{\mathcal{A}_H} = t^2 - 2t + 1$ and $\chi_{\mathcal{A}^H} = t - 2$.

Recursion for Deletion and Restriction

Theorem

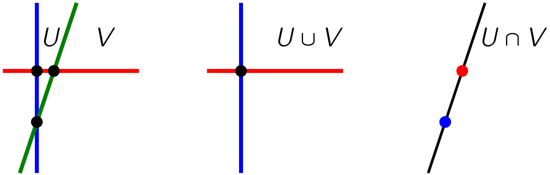
Let $\mathcal{A}$ be an arrangement and $H \in \mathcal{A}$, then

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}_H}(t) - \chi_{\mathcal{A}^H}(t)$$

Let $\mathrm{R}(\mathcal{A})$ be the number of regions described by $\mathcal{A}$.

Observation

For any arrangement $\mathcal{A}$ and $H \in \mathcal{A}$, we have $\mathrm{R}(\mathcal{A}) = \mathrm{R}(\mathcal{A}_H) + \mathrm{R}(\mathcal{A}^H)$.



Number of Regions

Example

Let $\mathcal{A}$ be the arrangement of k points in the real line $\mathbb{R}$, we have:

$$\chi_{\mathcal{A}}(t) = t - k, \mathrm{R}(\mathcal{A}) = k + 1 \text{ and } \mathrm{BR}(\mathcal{A}) = k - 1.$$

Theorem

The number of regions is $\mathrm{R}(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1)$.

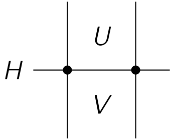
Proof.

The two functions $\mathrm{R}(\mathcal{A})$ and $(-1)^n \chi_{\mathcal{A}}(-1)$ satisfy the same recursion (deletion-restriction) and they coincide on all the arrangements in $\mathbb{R}$. □

Bounded Regions

Let $\mathrm{BR}(\mathcal{A})$ be the number of (relative) bounded regions, those number satisfy the recursion:

$$\mathrm{BR}(\mathcal{A}) = \begin{cases} \mathrm{BR}(\mathcal{A}_H) + \mathrm{BR}(\mathcal{A}^H) & \text{if } M(\mathcal{A}_H) \text{ does not contain any line} \\ 0 & \text{if } M(\mathcal{A}_H) \text{ contains a line} \end{cases}$$



Lemma

If $M(A_H)$ contains a line then $\chi_A(1) = 0$.

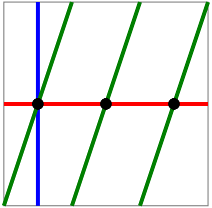
Theorem

The number of (relative) bounded regions is $\mathrm{BR}(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(1)$.

Definitions

Consider the compact torus $T^n := (S^1)^n$ and a finite collection $\mathcal{A}$ of subtori $H_1, \ldots, H_k$ of codimension one. We call $\mathcal{A}$ a *toric arrangement*.

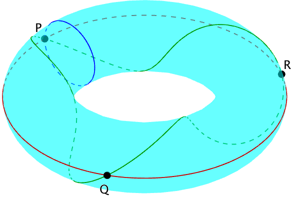
Example



$t_2 = 1$

$t_1^3 t_2 = 1$

$t_1 = 1$



Problem: How many open regions are described by the toric arrangement?

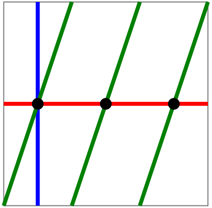
The intersection of some hypertori can be disconneted!

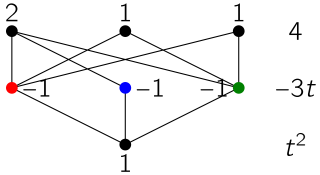
Poset and Characteristic Polynomial

We consider the poset $\mathcal{L}(\mathcal{A})$ of *connected components of intersections*.

Example

The Hasse diagram of the poset $\mathcal{L}(\mathcal{A})$ of the previous example is shown on the right, labelled with the Möbius function $\mu_{\mathcal{L}}(0, x)$ for $x \in \mathcal{L}(\mathcal{A})$.





The characteristic polynomial is $\chi_{\mathcal{A}}(t) = t^2 - 3t + 4$.

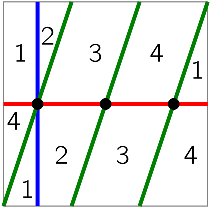
As done before, we define the Möbius function for the poset $\mathcal{L}(\mathcal{A})$ and the characteristic polynomial of the toric arrangement.

Toric Regions

We call $\mathrm{TR}(\mathcal{A})$ the number of toric regions described by $\mathcal{A}$.

Example

The running example describes 4 regions in the torus T^2.



Lemma

For $\mathcal{A}$ a toric arrangement and $H \in \mathcal{A}$ an hypertorus, we have

$$\mathrm{TR}(\mathcal{A}) = \begin{cases} \mathrm{TR}(\mathcal{A}_H) + \mathrm{TR}(\mathcal{A}^H) & \text{if } M(\mathcal{A}_H) \text{ does not contain any 1-torus} \\ \mathrm{TR}(\mathcal{A}^H) & \text{if } M(\mathcal{A}_H) \text{ contains a 1-torus} \end{cases}$$

Number of Toric Regions

Example

Let $\mathcal{A}$ be the toric arrangement of k points in the 1-torus S^1, we have:

$$\chi_{\mathcal{A}}(t) = t - k, \text{ and } \mathrm{TR}(\mathcal{A}) = k.$$

Lemma

If $M(\mathcal{A}_H)$ contains a 1-torus then $\chi_{\mathcal{A}_H}(0) = 0$.

Theorem

Let $\mathcal{A}$ be a toric arrangement, the number of toric regions is $\mathrm{TR}(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(0)$.

Proof.

The two functions $\mathrm{TR}(\mathcal{A})$ and $(-1)^n \chi_{\mathcal{A}}(0)$ satisfy the same recursion (deletion-restriction) and they coincide on all the arrangements in S^1. □

Thanks for listening!