Roberto Pagaria
Università di Bologna

Hodge theory for polymatroids

joint work with Gian Marco Pezzoli

at University of Strasbourg

May 18, 2022

Covered topics:

Polymatroids and subspace arrangements

Geometry and wonderful models

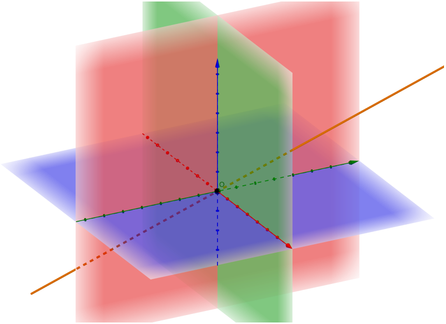
Leray model for polymatroids

The Kähler package

Subspace arrangements

Definition

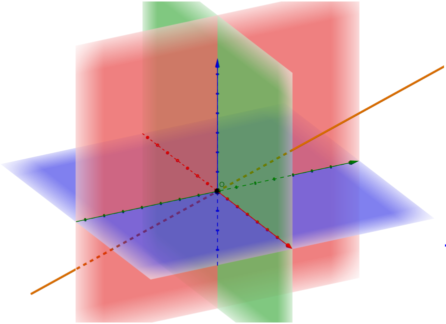
A *subspace arrangement* in a complex vector space V is a finite collection of linear subspaces S_i of V.

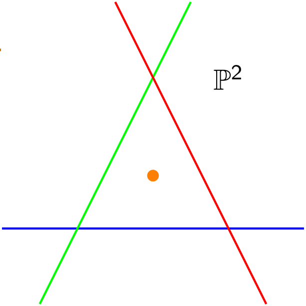


Subspace arrangements

Definition

A *subspace arrangement* in a complex vector space V is a finite collection of linear subspaces S_i of V.





Sometimes is useful to work with the projective version: the collection of $\mathbb{P}(S_i) \subset \mathbb{P}(V)$.

For $I \subseteq [n] = \{1, 2, \dots, n\}$ define the *codimension function* $\mathrm{cd}(I) = \mathrm{codim}_V(\cap_{i \in I} S_i)$ as the complex codimension of the *flat* $\cap_{i \in I} S_i$.

For $I \subseteq [n] = \{1, 2, \dots, n\}$ define the *codimension function* $\mathrm{cd}(I) = \mathrm{codim}_V(\cap_{i \in I} S_i)$ as the complex codimension of the *flat* $\cap_{i \in I} S_i$.

Example

In $\mathbb{C}^5$ consider S_a, S_b two subspace of dimension three and a line S_c in general position. We have $\mathrm{cd}(a) = 2, \mathrm{cd}(c) = 4$ and $\mathrm{cd}(ac) = \mathrm{cd}(bc) = \mathrm{cd}(abc) = 5$. Observe that $S_a \cap S_c = S_b \cap S_c$.

Polymatroids

A *polymatroid* P is a function $\mathrm{cd}\colon \mathcal{P}([n]) \to \mathbb{N}$ such that

1. $\mathrm{cd}(\emptyset) = 0$,
2. cd is increasing: $A \subset B$ implies $\mathrm{cd}(A) \leq \mathrm{cd}(B)$.
3. cd is submodular: $\mathrm{cd}(A) + \mathrm{cd}(B) \geq \mathrm{cd}(A \cap B) + \mathrm{cd}(A \cup B)$ for all A, B.

Polymatroids

A *polymatroid* P is a function $\mathrm{cd}\colon \mathcal{P}([n]) \to \mathbb{N}$ such that

1. $\mathrm{cd}(\emptyset) = 0$,
2. cd is increasing: $A \subset B$ implies $\mathrm{cd}(A) \leq \mathrm{cd}(B)$.
3. cd is submodular: $\mathrm{cd}(A) + \mathrm{cd}(B) \geq \mathrm{cd}(A \cap B) + \mathrm{cd}(A \cup B)$ for all A, B.

These objects codify the combinatorics of:

1. subspace arrangements,
2. cycles in an hypergraph,
3. generalized permutohedra.

A *flat* $F \subseteq [n]$ of codimension k is a maximal subset such that $\mathrm{cd}(F) = k$.

The poset of flats

Definition (Poset of flats)

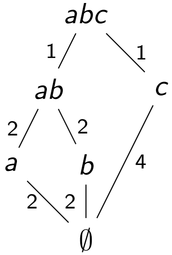
Let L be the set of all flats of the polymatroid P ordered by reverse inclusion.

The poset of flats

Definition (Poset of flats)

Let L be the set of all flats of the polymatroid P ordered by reverse inclusion.

Example



In general L is not a geometric lattice and is not ranked.

Wonderful model

The *complement* is $M = V \setminus (\cup_{i=1}^{n} S_i)$.

Wonderful model

The *complement* is $M = V \setminus (\cup_{i=1}^{n} S_i)$.

Definition

A *wonderful model* is a smooth projective variety Y containing M as open subset such that $Y \setminus M$ is a simple normal crossing divisor.

(simple normal crossing divisor: the irreducible components are smooth and intersect locally as coordinate hyperplanes)

Wonderful model

The *complement* is $M = V \setminus (\cup_{i=1}^{n} S_i)$.

Definition

A *wonderful model* is a smooth projective variety Y containing M as open subset such that $Y \setminus M$ is a simple normal crossing divisor.

(simple normal crossing divisor: the irreducible components are smooth and intersect locally as coordinate hyperplanes)
Let $\mathcal{G} \subset L$ be a "well chosen" collection of flats and consider

$$M \hookrightarrow V \times \bigtimes_{W \in \mathcal{G}} \mathbb{P}(V/W).$$

Let $Y_{\mathcal{G}}$ be the closure of the image of M.

Wonderful model

The *complement* is $M = V \setminus (\cup_{i=1}^{n} S_i)$.

Definition

A *wonderful model* is a smooth projective variety Y containing M as open subset such that $Y \setminus M$ is a simple normal crossing divisor.

(simple normal crossing divisor: the irreducible components are smooth and intersect locally as coordinate hyperplanes)
Let $\mathcal{G} \subset L$ be a "well chosen" collection of flats and consider

$$M \hookrightarrow V \times \bigtimes_{W \in \mathcal{G}} \mathbb{P}(V/W).$$

Let $Y_{\mathcal{G}}$ be the closure of the image of M.

Theorem (De Concini, Procesi '95)

The variety $Y_{\mathcal{G}}$ is a wonderful model for M.

Building sets

A subset $\mathcal{G}$ of L is a *building set* if for all $x \in L$

$$[\hat{0}, x] = \prod_{y \in \max(\mathcal{G}_{\leq x})} [\hat{0}, y]$$

and

$$\operatorname{cd}(x) = \sum_{y \in \max(\mathcal{G}_{\leq x})} \operatorname{cd}(y).$$

Building sets

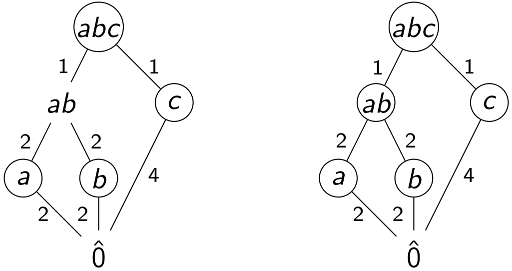
A subset $\mathcal{G}$ of L is a *building set* if for all $x \in L$

$$[\hat{0}, x] = \prod_{y \in \max(\mathcal{G}_{\leq x})} [\hat{0}, y]$$

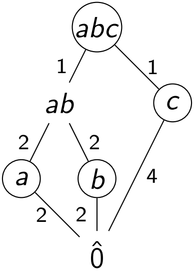
and

$$\mathrm{cd}(x) = \sum_{y \in \max(\mathcal{G}_{\leq x})} \mathrm{cd}(y).$$

Example



Example



If $\mathcal{G} = \{abc, a, b, c\}$ is the minimal building set of the previous example. Then the wonderful model is $Y_{\mathcal{G}} = \mathrm{Bl}_{S_a} \mathrm{Bl}_{S_b} \mathrm{Bl}_{S_c} \mathrm{Bl}_0 \mathbb{C}^5$ a sequence of blow-ups.

$\mathcal{G}$-nested sets

The simple normal crossing divisor $Y_{\mathcal{G}} \setminus M$ has irreducible components $\{D_W\}_{W \in \mathcal{G}}$ in bijections with the building set $\mathcal{G}$.

$\mathcal{G}$-nested sets

The simple normal crossing divisor $Y_{\mathcal{G}} \setminus M$ has irreducible components $\{D_W\}_{W \in \mathcal{G}}$ in bijections with the building set $\mathcal{G}$.

Definition

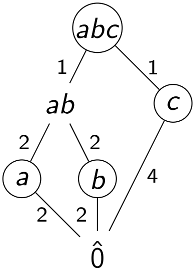
A set $S \subseteq \mathcal{G}$ is *$\mathcal{G}$-nested* if the intersection $\cap_{W \in S} D_W$ is non-empty.
Abstractly, $S \subseteq \mathcal{G}$ is *$\mathcal{G}$-nested* if for any non-trivial antichain $T \in S$ we have $\bigvee T \notin \mathcal{G}$.

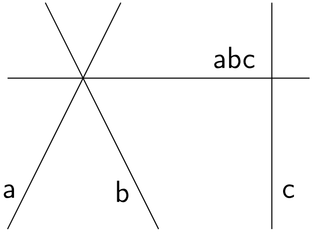
$\mathcal{G}$-nested sets

The simple normal crossing divisor $Y_{\mathcal{G}} \setminus M$ has irreducible components $\{D_W\}_{W \in \mathcal{G}}$ in bijections with the building set $\mathcal{G}$.

Definition

A set $S \subseteq \mathcal{G}$ is $\mathcal{G}$*-nested* if the intersection $\cap_{W \in S} D_W$ is non-empty. Abstractly, $S \subseteq \mathcal{G}$ is $\mathcal{G}$*-nested* if for any non-trivial antichain $T \in S$ we have $\bigvee T \notin \mathcal{G}$.

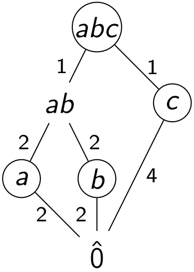


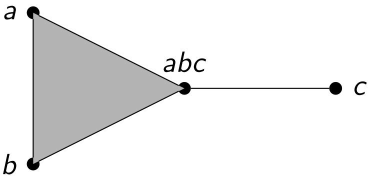


Nested set complex

Let $n(\mathcal{G})$ be the collection of all $\mathcal{G}$-nested sets. It is an *abstract simplicial complex*.

Example





Previous works

- De Concini, Procesi '95 described the Chow ring $A(Y_{\mathcal{G}})$ (cohomology) of wonderful models.
- Feichtner, Yuzvinsky '03 described the Chow ring $A(L)$ of an atomic lattice with a building set.
- Huh, Adiprasito, Katz '18 proved the Kähler package for $A(L)$ of a geometric lattice with the maximal building set.

Previous works

- De Concini, Procesi '95 described the Chow ring $A(Y_{\mathcal{G}})$ (cohomology) of wonderful models.
- Feichtner, Yuzvinsky '03 described the Chow ring $A(L)$ of an atomic lattice with a building set.
- Huh, Adiprasito, Katz '18 proved the Kähler package for $A(L)$ of a geometric lattice with the maximal building set.
- De Concini, Procesi '95 described the Leray model $B(\mathcal{G})$ for $M \hookrightarrow Y_{\mathcal{G}}$.
- Yuzvinsky '02, '99 simplified the model of De Concini Procesi and relates it to the Goresky-MacPherson formula.
- Bibby, Denham, Feichtner '21 studied the Leray model $B(\mathcal{G})$ for geometric lattices and partial building sets.

Leray model and Chow ring

The *Leray model* $(B^{\cdot,\cdot}(\mathcal{G}), \mathrm{d})$ is the second page of the Leray spectral sequence for $M \hookrightarrow Y_{\mathcal{G}}$ (aka the Morgan algebra). Furthermore, $B^{\cdot,0}(\mathcal{G}) = H^{\cdot}(Y_{\mathcal{G}}) = A^{\cdot}(Y_{\mathcal{G}})$ and $H^{\cdot}(B(\mathcal{G}), \mathrm{d}) = H^{\cdot}(M)$.

Leray model and Chow ring

The *Leray model* $(B^{\cdot,\cdot}(\mathcal{G}), \mathrm{d})$ is the second page of the Leray spectral sequence for $M \hookrightarrow Y_{\mathcal{G}}$ (aka the Morgan algebra). Furthermore, $B^{\cdot,0}(\mathcal{G}) = H^{\cdot}(Y_{\mathcal{G}}) = A^{\cdot}(Y_{\mathcal{G}})$ and $H^{\cdot}(B(\mathcal{G}), \mathrm{d}) = H^{\cdot}(M)$.

Explicitly, $B^{\cdot,\cdot}(\mathcal{G})$ is generated by e_W, x_W for $W \in \mathcal{G}$ with bidegree $(0,1)$ and $(2,0)$ respectively and relations:

- $e_T x_S (\sum_{Z \geq W} x_Z)^b = 0$ for $S, T \subset \mathcal{G}$, $W \in \mathcal{G}$ and $b = \mathrm{cd}(W) - \mathrm{cd}(\bigvee (T \cup S)_{<W})$,

with differential defined by $\mathrm{d}(e_W) = x_W$.

(we use the notation $e_T = \prod_{W \in T} e_W$.)

Leray model and Chow ring

The *Leray model* $(B^{\cdot,\cdot}(\mathcal{G}), \mathrm{d})$ is the second page of the Leray spectral sequence for $M \hookrightarrow Y_{\mathcal{G}}$ (aka the Morgan algebra). Furthermore, $B^{\cdot,0}(\mathcal{G}) = H^{\cdot}(Y_{\mathcal{G}}) = A^{\cdot}(Y_{\mathcal{G}})$ and $H^{\cdot}(B(\mathcal{G}), \mathrm{d}) = H^{\cdot}(M)$.

Explicitly, $A^{\cdot}(\mathcal{G})$ is generated by x_W for $W \in \mathcal{G}$ of degree 1 and relations:

- $x_S(\sum_{Z \geq W} x_Z)^b = 0$ for $S \subset \mathcal{G}$, $W \in \mathcal{G}$ and $b = \mathrm{cd}(W) - \mathrm{cd}(\bigvee(S_{<W}))$.

In the realizable case $x_W = [D_W]$ is the fundamental class of the (exceptional) divisor associated to W.

A second presentation

Define $\sigma_W = \sum_{Z \geq W} x_Z$ and $\tau_W = \sum_{Z \geq W} e_Z$. Geometrically, $\sigma_W \in A^1(Y_{\mathcal{G}})$ is the fundamental class of the total transform of W:

$$\sigma_W = [\pi^{-1}(W)],$$

where $\pi \colon Y_{\mathcal{G}} \to \mathbb{P}(V)$ is the canonical projection.

A second presentation

Define $\sigma_W = \sum_{Z \geq W} x_Z$ and $\tau_W = \sum_{Z \geq W} e_Z$. Geometrically, $\sigma_W \in A^1(Y_{\mathcal{G}})$ is the fundamental class of the total transform of W:

$$\sigma_W = [\pi^{-1}(W)],$$

where $\pi\colon Y_{\mathcal{G}} \to \mathbb{P}(V)$ is the canonical projection.
The Leray model $B^{\cdot,\cdot}(\mathcal{G})$ is generated by τ_W, σ_W for $W \in \mathcal{G}$ with bidegree $(0, 1)$ and $(2, 0)$ respectively and relations:

- $\prod_{t \in T}(\tau_t - \tau_W) \prod_{t \in S}(\sigma_t - \sigma_W)\sigma_W^b = 0$ for $S, T \subset \mathcal{G}$, $W \in \mathcal{G}$ and $b = \operatorname{cd}(W) - \operatorname{cd}(\bigvee(T \cup S)_{<W})$,

with differential defined by $\mathrm{d}(\tau_W) = \sigma_W$.

Goresky MacPherson formula

Consider a subspace arrangement with complement M and poset of flats L.

Theorem (Goresky MacPherson '88)

There is an additive isomorphism

$$\tilde{H}^k(M;\mathbb{Z}) \cong \bigoplus_{W \in L \setminus \hat{0}} \tilde{H}_{2\,\mathrm{cd}(W)-2-k}(\Delta((\hat{0}, W)); \mathbb{Z}),$$

where $\Delta((\hat{0}, W))$ is the order complex of the interval $(\hat{0}, W)$.

We used the convention that $\tilde{H}_{-1}(\emptyset, \mathbb{Z}) = \mathbb{Z}$.

The critical monomial algebra

Theorem (Yuzvinsky '99, P. Pezzoli '21)

There exists a critical monomial algebra $\mathrm{CM}(\mathcal{G}) \subset B(\mathcal{G})$ *such that the inclusion is a quasi-isomorphism.*

The critical monomial algebra

Theorem (Yuzvinsky '99, P. Pezzoli '21)

There exists a critical monomial algebra $\mathrm{CM}(\mathcal{G}) \subset B(\mathcal{G})$ *such that the inclusion is a quasi-isomorphism. Moreover,*

$$\tilde{H}^\bullet(\mathrm{CM}(\mathcal{G}), \mathrm{d}) \cong \bigoplus_{W \in L \setminus \hat{0}} \bigotimes_{Z \in \max(\mathcal{G}_{\leq W})} \tilde{H}_{2\,\mathrm{cd}(Z)-2-\bullet}(n(\mathcal{G}, Z)),$$

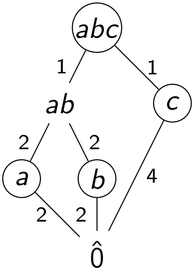
The critical monomial algebra

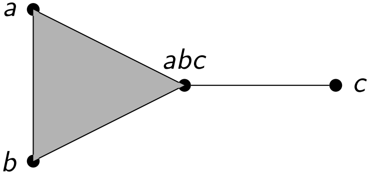
Theorem (Yuzvinsky '99, P. Pezzoli '21)

There exists a critical monomial algebra $\mathrm{CM}(\mathcal{G}) \subset B(\mathcal{G})$ *such that the inclusion is a quasi-isomorphism. Moreover,*

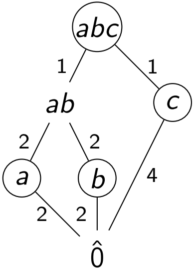
$$\tilde{H}^{\bullet}(\mathrm{CM}(\mathcal{G}), \mathrm{d}) \cong \bigoplus_{W \in L \setminus \hat{0}} \bigotimes_{Z \in \max(\mathcal{G}_{\leq W})} \tilde{H}_{2\,\mathrm{cd}(Z)-2-\bullet}(n(\mathcal{G}, Z)),$$

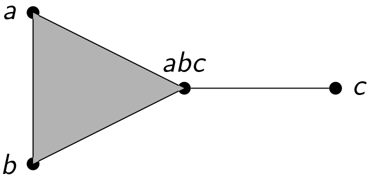
where $n(\mathcal{G}, Z)$ *is the* $\mathcal{G}$*-nested set complex of* $(\hat{0}, Z)$*.*





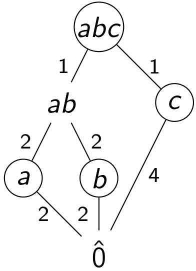
$$\tilde{H}^\bullet(\mathrm{CM}(\mathcal{G}), \mathrm{d}) \cong \bigoplus_{W \in L \setminus \hat{0}} \bigotimes_{Z \in \max(\mathcal{G}_{\leq W})} \tilde{H}_{2\,\mathrm{cd}(Z)-2-\bullet}(n(\mathcal{G}, Z)),$$





addendum	hom degree	degree	W
$\mathbb{Z}$	0	8	abc
$\mathbb{Z}$	-1	3	a
$\mathbb{Z}$	-1	3	b
$\mathbb{Z}$	-1	7	c
$\mathbb{Z}$	-1	6	ab

$$\tilde{H}^\bullet(\mathrm{CM}(\mathcal{G}), \mathrm{d}) \cong \bigoplus_{W \in L \setminus \hat{0}} \bigotimes_{Z \in \max(\mathcal{G}_{\leq W})} \tilde{H}_{2\,\mathrm{cd}(Z)-2-\bullet}(n(\mathcal{G}, Z)),$$



addendum	hom degree	degree	W
$\mathbb{Z}$	0	8	abc
$\mathbb{Z}$	-1	3	a
$\mathbb{Z}$	-1	3	b
$\mathbb{Z}$	-1	7	c
$\mathbb{Z}$	-1	6	ab

Definitions

Let A be an algebra with top degree n and $\deg\colon A^n \to \mathbb{Q}$ an isomorphism.

- the algebra A satisfies *Poincaré duality* if the bilinear pairing
$$A^k \times A^{n-k} \to \mathbb{Q}$$
defined by $(a, b) \mapsto \deg(ab)$ is non-degenerate.

Definitions

Let A be an algebra with top degree n and $\deg\colon A^n \to \mathbb{Q}$ an isomorphism.

- the algebra A satisfies *Poincaré duality* if the bilinear pairing
$$A^k \times A^{n-k} \to \mathbb{Q}$$
defined by $(a, b) \mapsto \deg(ab)$ is non-degenerate.
- the element $\ell \in A^1$ satisfies the *Hard Lefschetz property* if
$$\cdot\ell^{n-2k}\colon A^k \to A^{n-k}$$
is an isomorphism for all $k \leq \frac{n}{2}$.

Definitions

Let A be an algebra with top degree n and $\deg\colon A^n \to \mathbb{Q}$ an isomorphism.

- the algebra A satisfies *Poincaré duality* if the bilinear pairing
$$A^k \times A^{n-k} \to \mathbb{Q}$$
defined by $(a, b) \mapsto \deg(ab)$ is non-degenerate.
- the element $\ell \in A^1$ satisfies the *Hard Lefschetz property* if
$$\cdot\ell^{n-2k}\colon A^k \to A^{n-k}$$
is an isomorphism for all $k \leq \frac{n}{2}$.
- the element $\ell \in A^1$ satisfies the *Hodge Riemann relations* if
$$Q_\ell^k\colon A^k \times A^k \to \mathbb{Q}$$
defined by $Q_\ell^k(a, b) = (-1)^k \deg(a\ell^{n-2k}b)$ (for $k \leq \frac{n}{2}$) is positive definite on the subspace
$$P_k = \ker(\cdot\ell^{n-2k+1}\colon A^k \to A^{n-k+1}).$$

Let L be a geometric lattice with $\mathrm{cd} = \mathrm{rk}$ and $\mathcal{G}$ be the maximal building set. The algebra $A(\mathcal{G})$ is the Chow ring of the matroid.

Theorem (Adiprasito, Huh, Katz '18)

The ring $A(\mathcal{G})$ is a Poincaré duality algebra and each $\ell = \sum_{W \neq \hat{1}} c_W x_W \in A^1(\mathcal{G})$ (ample) such that

$$c_W + c_Z > c_{W \vee Z} + c_{W \wedge Z}$$

satisfies Hard Lefschetz and Hodge Riemann relations.

Let L be a geometric lattice with $\mathrm{cd} = \mathrm{rk}$ and $\mathcal{G}$ be the maximal building set. The algebra $A(\mathcal{G})$ is the Chow ring of the matroid.

Theorem (Adiprasito, Huh, Katz '18)

The ring $A(\mathcal{G})$ is a Poincaré duality algebra and each $\ell = \sum_{W \neq \hat{1}} c_W x_W \in A^1(\mathcal{G})$ (ample) such that

$$c_W + c_Z > c_{W \vee Z} + c_{W \wedge Z}$$

satisfies Hard Lefschetz and Hodge Riemann relations.

The Hodge Riemann relations prove a conjecture by Read, Hoggar, Rota, Heron, Welsh '60s-'70s:

Corollary (Adiprasito, Huh, Katz '18)

The coefficients of the characteristic polynomial for a log-concave sequence.

Let L be the poset of flats of a polymatroid and $\mathcal{G}$ an arbitrary building set.

Theorem (P. Pezzoli '21)

The ring $A(\mathcal{G})$ is a Poincaré duality algebra and each $\ell = -\sum_{W \in \mathcal{G}} d_W \sigma_W \in A^1(\mathcal{G})$ such that

$$d_W > 0$$

satisfies Hard Lefschetz and Hodge Riemann relations.

Let L be the poset of flats of a polymatroid and $\mathcal{G}$ an arbitrary building set.

Theorem (P. Pezzoli '21)

The ring $A(\mathcal{G})$ is a Poincaré duality algebra and each $\ell = -\sum_{W \in \mathcal{G}} d_W \sigma_W \in A^1(\mathcal{G})$ such that

$$d_W > 0$$

satisfies Hard Lefschetz and Hodge Riemann relations.

We call this orthant the σ-cone.

Remark

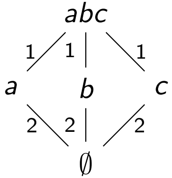
The σ-cone is contained in the ample cone of any realization, but for polymatroids the ample cone depends on the chosen realization.

Remark

The σ-cone is contained in the ample cone of any realization, but for polymatroids the ample cone depends on the chosen realization.

Example

Consider the polymatroid realized by three distinct lines in $\mathbb{C}^3$.



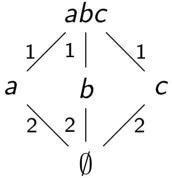
$Y_{\mathcal{G}}$ is the blowup of $\mathbb{P}^2$ in three points. If the three points are in general position then the ample cone coincides with the σ-cone.

Remark

The σ-cone is contained in the ample cone of any realization, but for polymatroids the ample cone depends on the chosen realization.

Example

Consider the polymatroid realized by three distinct lines in $\mathbb{C}^3$.



$Y_{\mathcal{G}}$ is the blowup of $\mathbb{P}^2$ in three points. If the three points are in general position then the ample cone coincides with the σ-cone.Otherwise the three points are collinear and the ample cone is given by:

$$\{-d_{abc}\sigma_{abc} - d_a\sigma_a - d_b\sigma_b - d_c\sigma_c \mid d_a, d_b, d_c > 0,$$
$$d_{abc} > -\min(d_a, d_b, d_c)\}$$

Remark

There are examples of polymatroids with (reduced) characteristic polynomial with negative coefficients and that do not form a log-concave sequence.

Remark

There are examples of polymatroids with (reduced) characteristic polynomial with negative coefficients and that do not form a log-concave sequence.

Remark

The main problem is that $x_{\hat{1}}$ behaves different from x_W for $W \in \mathcal{G} \setminus \hat{1}$.

Sketch of the proof

Theorem (P. Pezzoli '21)

The Chow ring of a polymatroid satisfies the Kähler package.

Sketch of the proof:

1. Present a Gröbner basis for $A(\mathcal{G})$,

Sketch of the proof

Theorem (P. Pezzoli '21)

The Chow ring of a polymatroid satisfies the Kähler package.

Sketch of the proof:

1. Present a Gröbner basis for $A(\mathcal{G})$,
2. Prove Poincaré duality constructing an explicit pairing,

Sketch of the proof

Theorem (P. Pezzoli '21)

The Chow ring of a polymatroid satisfies the Kähler package.

Sketch of the proof:

1. Present a Gröbner basis for $A(\mathcal{G})$,
2. Prove Poincaré duality constructing an explicit pairing,
3. Prove recursive relations using Poincaré duality,

Sketch of the proof

Theorem (P. Pezzoli '21)

The Chow ring of a polymatroid satisfies the Kähler package.

Sketch of the proof:

1. Present a Gröbner basis for $A(\mathcal{G})$,
2. Prove Poincaré duality constructing an explicit pairing,
3. Prove recursive relations using Poincaré duality,
4. Prove simultaneously Hard Lefschetz and Hodge Riemann by induction on $|\mathcal{G}|$.

Thanks for listening!

roberto.pagaria@unibo.it