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The S_n-action on the Orlik-Terao algebra of type A_{n-1}

Arrangements in Ticino

June, 2022

A short story:

Moseley-Proudfoot-Young conjecture

Failing approaches

Orlik-Terao algebra of type A_n

Once upon a time

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Proudfoot, Ramos, *Stability phenomena for resonance arrangements*.

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in which they report a conjecture of

Moseley, Proudfoot, Young *The Orlik-Terao algebra and the cohomology of configuration space*

The two dragons

Definition (M_n)

Is the algebra $\mathbb{Q}[x_{i,j}]_{i\neq j}$ with relations

1. $x_{i,j} + x_{j,i}$
2. $\sum_{j\neq i} x_{i,j}$ for all $i = 1, \ldots, n$
3. $x_{i,j}x_{j,k} + x_{j,k}x_{k,i} + x_{k,i}x_{i,j}$

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$$\sigma(x_{i,j}) = x_{\sigma(i),\sigma(j)}$$

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Conjecture (Moseley-Proudfoot-Young '16)

There exists an isomorphism of graded S_n-representations

$$M_n \simeq_{S_n} D_n.$$

The geometry of the D_n dragon's den

Let SU_2 be the special unitary group

$$SU_2 = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\} \simeq S^3$$

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Proposition

The algebra D_n is the cohomology of $\mathrm{Conf}_n(SU_2)/SU_2$

$$D_n \simeq H^{2\cdot}(\mathrm{Conf}_n(SU_2)/SU_2; \mathbb{Q})$$

where SU_2 acts freely by group multiplication.

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D_n^i is the *Whitehouse representation*, the top grade D_n^{n-2} is

- the multilinear part of the free Lie algebra Lie_{n-1},
- the homology of nonmodular partitions,
- the homology of homeomorphically irreducible trees...

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Let $T^a = (\mathbb{C}^*)^a$ be an algebraic torus. Consider

$$T^{\binom{n}{2}} \to T^{n-1} \to 0$$

$$x_{i,j} \mapsto z_i z_j^{-1}$$

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$$X_n = T^*\mathbb{C}^{\binom{n}{2}} /\!\!/\!\!/\!\!/_0 \, T^k$$

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Theorem (Braden Proudfoot '09)

There exists an S_n-isomorphism of graded ring

$$M_n \simeq IH^{2\cdot}(X_n; \mathbb{Q}).$$

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Conjecture (Moseley-Proudfoot-Young '16)

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- $M_n = D_n$ for $n \leq 22$ using SageMath (Matherne, Miyata, Proudfoot, Ramos '21)
- $M_n^i = D_n^i$ is true for $i \leq 7$ using representation stability (MMPR '21)

The early January attack

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Q: Are M_n and D_n two presentation of the same algebra?

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Remark

The algebras M_n and D_n are not isomorphic for $n \geq 5$.

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The family $\mathbb{Q}[x_{i,j}, t]_{i \neq j}$ with relations

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is equal to M_n for $t = 0$ and to D_n for $t = 1$. However, SageMath shows that these two points are not in the flat locus for $n \geq 5$.

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Q: Is there any deformation between M_n and D_n?

Fact 1: The irreducible representation of S_n are parametrized by the *partition* of n. Let $\mathbb{V}_\lambda$ be the irreducible representation associated with $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\sum_i \lambda_i = n$.

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Fact 2: There exists a **Frobenius characteristic** function

$$\mathrm{ch} : \{S_n \text{ representations}\} \to \{\text{symmetric polynomials}\}$$
$$\mathbb{V}_\lambda \longmapsto s_\lambda$$

where s_λ is the *Schur symmetric* polynomial.

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\begin{aligned}
M_n^1 &= D_n^1 = \mathbb{V}_{n-2,1,1} \\
S^2\mathbb{V}_{n-2,1,1} &= \mathbb{V}_{n-4,1,1,1,1} + \mathbb{V}_{n-4,2,2} + \mathbb{V}_{n-3,2,1} \\
&\quad + \mathbb{V}_{n-3,3} + 2\mathbb{V}_{n-2,2} + \mathbb{V}_{n-1,1} + \mathbb{V}_n \\
M_n^2 &= D_n^2 = \mathbb{V}_{n-4,1,1,1,1} + \mathbb{V}_{n-4,2,2} + \mathbb{V}_{n-3,2,1} + \mathbb{V}_{n-2,2}
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hence there is a unique graded S_n-equivariant "deformation" between M_n and D_n with the desired properties in degree one and two. It is parametrized by $\mathbb{P}^1$.

The first assistant C_n

Goal: describe D_n as graded S_n-representation.

Definition (Artinian Orlik-Terao algebra of type A_n)

Define C_n as the quotient of $\mathbb{Q}[x_{i,j}]_{i \neq j}$ by the relations

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Theorem (Cohen '76)

The cohomology of the configuration space on $\mathbb{R}^3$ is

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- $\mathrm{Res}_{S_n}^{S_{n+1}} D_{n+1} = C_n$,
- $C_n = D_n \otimes (\mathbb{V}_n + q\mathbb{V}_{n-1,1})$.

Theorem (Sundaram, Welker '97)

As S_n-representation

$$C_n^i = \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda) = n-i}} \mathrm{Ind}_{\times_i Z_i \wr S_{m_i}}^{S_n} \boxtimes \zeta_i$$

where $\lambda = (1^{m_1}, 2^{m_2}, \dots)$.

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The graded Frobenius characteristic of C_n *is*

$$\operatorname{ch} C_n = \sum_{\lambda \vdash n} q^{\sum_i (i-1) m_i} \prod_{i=1}^{n} h_{m_i}[\ell_i],$$

where $\ell_i = \operatorname{ch} \operatorname{Ind}_{Z_i}^{S_i} \zeta_i$ *is the Lyndon/Gessel-Reutenauer symmetric function, ie the character of the multilinear component of the free Lie algebra.*

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The *plethysm* $f[g]$ is an operation on symmetric functions such that

$$\mathrm{ch}\, W[\mathrm{ch}\, V] = \mathrm{ch}\,\mathrm{Ind}^{S_{hk}}_{S_k \wr S_h}(V^{\boxtimes h} \otimes W)$$

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Definition

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Theorem (Moseley, Proudfoot, Young '16)

The following holds:

$$M_n \otimes R_n = \bigoplus_{\lambda \vdash n} \mathrm{Ind}^{S_n}_{\times S_i \wr S_{m_i}} \left(C_{l(\lambda)} \otimes \boxtimes_i (M_i^c \otimes R_i) \boxtimes (\mathbb{V}_{m_i} \oplus \mathbb{V}_{m_i-1,1}) \right)$$

where $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ *and* $(M_n^c)^i = M_n^{2n-2-i}$.

The late January attack

The representation D_n can be described from

$$C_n = D_n \otimes (\mathbb{V}_n + q\mathbb{V}_{n-1,1}),$$

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The MPY-conjecture follows by showing that D_n satisfies the recurrence relation

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The Frobenius characteristic of D_n can be described from

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$$\begin{aligned} &\operatorname{ch} M_n * \operatorname{ch} R_n = \\ &= \sum_{\substack{\nu_1, \ldots \nu_n \\ \sum_i i\nu_i = n}} \langle s_{\nu_1} \ldots s_{\nu_n}, \operatorname{ch} C_{\sum \nu_i} \rangle \prod_i s_{\nu_i} \left[q^{2i-2} (\operatorname{ch} M_i)_{|q=q^{-1}} * \operatorname{ch} R_i \right] \end{aligned}$$

where

$$\operatorname{ch} R_n = (1-q) h_n \left[\frac{X}{1-q} \right].$$

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The MPY-conjecture follows by showing that $\operatorname{ch} D_n$ satisfies the recurrence relation

$$\operatorname{ch} M_n * \operatorname{ch} R_n =$$

$$= \sum_{\substack{\nu_1, \dots \nu_n \\ \sum_i i\nu_i = n}} \langle s_{\nu_1} \dots s_{\nu_n}, \operatorname{ch} C_{\sum \nu_i} \rangle \prod_i s_{\nu_i} \left[q^{2i-2} (\operatorname{ch} M_i)_{|q=q^{-1}} * \operatorname{ch} R_i \right]$$

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Idea: use spectral sequence to compute the S_n action on D_n.

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Theorem (Cohen, Taylor '78)

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$$E_r^{p,q}(SU_2, n) \Rightarrow H^{p+q}(\mathrm{Conf}_n(SU_2); \mathbb{Q})$$

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The February attack

Idea: use spectral sequence to compute the S_n action on D_n.

Theorem (Cohen, Taylor '78)

There exists a spectral sequence

$$E_r^{p,q}(SU_2, n) \Rightarrow H^{p+q}(\mathrm{Conf}_n(SU_2); \mathbb{Q})$$

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It degenerates at the fourth page by comparison with the SS $E_r^{p,q}(\mathbb{R}^3, n) \Rightarrow H^{p+q}(\mathrm{Conf}_n(\mathbb{R}^3))$.

The February attack

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$$\begin{array}{c|cccc}
4 & * & * & & \\
2 & * & * & * & \\
0 & * & * & * & * \\
\hline
 & 0 & 3 & 6 & 9
\end{array}$$

$E_2(SU_2, 3)$

The February attack

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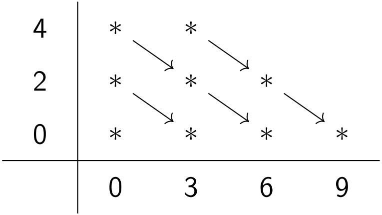

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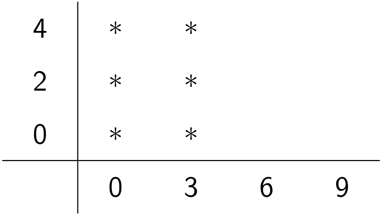

$E_\infty(SU_2, 3)$

We also used

$$H^\cdot(\mathrm{Conf}_n(SU_2); \mathbb{Q}) = H^\cdot(\mathrm{Conf}_n(SU_2)/SU_2; \mathbb{Q}) \otimes H^\cdot(SU_2; \mathbb{Q})$$

by the Leray-Hirsch theorem.

Defeating the drake D_n

By taking the "refined" Euler characteristic of $E_2^{p,q}(SU_2, n)$ and of $E_\infty^{p,q}(SU_2, n)$ we obtain:

Corollary (P. '22)

The Frobeinus characteristic of D_n is

$$\begin{aligned} \operatorname{ch} D_n &= \sum_{\lambda \vdash n} \frac{q^{n-l(\lambda)}}{1-q} \prod_i h_{m_i}[(1-q)\ell_i] \\ &= \frac{1}{1-q} \sum_{\lambda \vdash n} \prod_i h_{m_i}[q^{i-1}(1-q)\ell_i], \end{aligned}$$

where $\lambda = (1^{m_1}, 2^{m_2}, \dots)$.

The return of the King

Definition (Terao '02)

The Orlik-Terao algebra OT_n of type A_n is the quotient of $\mathbb{Q}[x_{i,j}]_{i\neq j}$ by the relations

1. $x_{i,j} + x_{j,i}$
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- OT_n degenerates flatly to the *Stanley-Reisner ring* of the *broken circuit complex*.

Defeating the drake M_n

Goal: a non-recursive description of M_n.

Theorem (Braden, Proudfoot '09)

They prove

$$M_n = IH^{2\cdot}(X_n;\mathbb{Q}),$$
$$R_n = IH^{2\cdot}_{T^{n-1}}(*;\mathbb{Q}),$$
$$OT_n = IH^{2\cdot}_{T^{n-1}}(X_n;\mathbb{Q}).$$

Hence, $OT_n = M_n \otimes R_n$ *as graded* S_n*-representations.*

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The King's cohort

We filter OT_n by the *support* of the monomials. Let T_n be the submodule of OT_n generated by monomials with full support.

Lemma (P. '22)

We have

$$OT_n = \bigoplus_{S \vdash [n]} \bigotimes_i T_{S_i}$$

and

$$\mathrm{ch}\, OT_n = \sum_{\lambda \vdash n} \prod_i h_{m_i}[\mathrm{ch}\, T_i].$$

The final battle

Theorem (P. '22)

We have

$$\operatorname{ch} M_n = \operatorname{ch} D_n,$$

$$\operatorname{ch} T_n = q^{n-1}\ell_n * \operatorname{ch} R_n = (1-q)q^{n-1}\ell_n * h_n\left[\frac{X}{1-q}\right].$$

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Proof.

By a simultaneous induction:

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The final battle

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and so

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□

The treasure

Corollary (P. '22)

We have

$$\mathrm{ch}\, OT_n = \sum_{\lambda \vdash n} q^{n-l(\lambda)} \prod_i h_{m_i}[\ell_i * \mathrm{ch}\, R_i].$$

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Corollary

The generating functions are

$$\sum_{n\geq 1} \mathrm{ch}_{D_n}(q) t^n = \sum_{n\geq 1} \mathrm{ch}_{M_n}(q) t^n = \frac{1}{1-q}(\mathrm{Exp}((1-q)L) - 1),$$

$$\sum_{n\geq 1} \mathrm{ch}_{OT_n}(q) t^n = \mathrm{Exp}\left((1-q)L * \mathrm{Exp}\left(\frac{X}{1-q}\right)\right) - 1.$$

...and they lived happily ever after

Proudfoot gave to his PhD student Moseley the problem of computing $\operatorname{ch} OT_n$ in 2008. He hasn't solved it, but long after they come up with the MPY conjecture $D_n = M_n$.
Finally, the circle is closed.

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full story in *The Frobenius characteristic of the Orlik-Terao algebra of type A* arXiv:2203.08265 March 15th, 2022
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to be continued...

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- The MPY conjecture is stated also for *graphical arrangements* $D_\Gamma \simeq M_\Gamma$ as graded representations of $\mathrm{Aut}(\Gamma)$, but we cannot use symmetric function!

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- The MPY conjecture is stated also for *graphical arrangements* $D_\Gamma \simeq M_\Gamma$ as graded representations of $\mathrm{Aut}(\Gamma)$, but we cannot use symmetric function!
- Does a similar statement holds for finite Coxeter arrangements? How to define D_W? maybe D_{B_n} is the cohomology of an *orbit configuration space*.

Contact me if you are interested!

The end

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