ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA
DIPARTIMENTO DI MATEMATICA

Cohomology rings of abelian arrangements

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Joint with E. Bazzocchi and M. Pismataro

Hyperplane Arrangements 2023, Tokyo

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Covered topics:

(a, b)-arrangements

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Example

(0,1) real hyperplane arrangement $G = \mathbb{R}$,

(0,2) complex hyperplane arrangement $G = \mathbb{C}$,

(1,1) toric arrangement $G = \mathbb{C}^*$.

Goal: describe the cohomology ring of the *complement* $M_{\mathcal{A}}^{a,b} := G^r \setminus \bigcup_{\alpha \in \mathcal{A}} H_\alpha$.

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For every i the map $\alpha_i \colon G^r \to G$ restricts to
$$\alpha_i \colon M_{\mathcal{A}}^{a,b} \to G \setminus \{e\}$$
and for any class $\omega \in H^*(G \setminus \{e\})$ we define
$$\omega_i := \alpha_i^*(\omega) \in H^*(M_{\mathcal{A}}^{a,b}).$$

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Hope: the classes ω_i generate $H^*(M_{\mathcal{A}}^{a,b})$ and it is possible to describe the relations between them.

Complex hyperplane arrangements

We consider the case $(0,2)$ of complex hyperplane arrangements.
The cohomology of $\mathbb{C} \setminus \{0\}$ is generated by the form $\omega = \frac{1}{2\pi\sqrt{-1}} \operatorname{d}\log z$ and

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Theorem (Orlik-Solomon 1980)

$$H^*(M_{\mathcal{A}}^{0,2}; \mathbb{Z}) \simeq {}^{\mathbb{Z}[\omega_1, \omega_2, \ldots, \omega_n]}\big/_{(\partial \omega_C \mid C \text{ circuit})}$$

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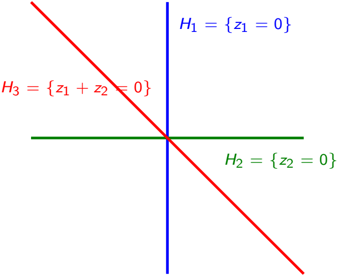
For each *circuit* C (i.e. minimal dependent set) of cardinality $k = |C|$ the Arnold relation is

$$\partial\omega_C := \sum_{i=1}^{k} (-1)^i \omega_{c_1}\omega_{c_2}\dots\widehat{\omega_{c_i}}\dots\omega_{c_k} = 0$$

and it holds at level of differential forms.

Example

Consider the complex arrangement in $\mathbb{C}^2$ given by the hyperplanes $H_1 = \{z_1 = 0\}, H_2 = \{z_2 = 0\}, H_3 = \{z_1 + z_2 = 0\}$.



There is a unique circuit $\{1, 2, 3\}$ and the associated Arnold relation is

$$\omega_2\omega_3 - \omega_1\omega_3 + \omega_1\omega_2 = 0.$$

Real hyperplane arrangements

The cohomology of $\mathbb{R} \setminus \{0\}$ is generated by the functions $\omega^+ = \delta_{\mathbb{R}^+}$ and $\omega^- = \delta_{\mathbb{R}^-}$ with relations $\omega^+ + \omega^- = 1$ and $\omega^+\omega^- = 0$.

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Theorem (Gelfand-Varchenko 1986)

The cohomology ring $H^*(M_{\mathcal{A}}^{0,1};\mathbb{Z}) = H^0(M_{\mathcal{A}}^{0,1};\mathbb{Z}) = \mathbb{Z}^{\pi_0(M_{\mathcal{A}}^{0,1})}$

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1. $\omega_i^+ + \omega_i^- = 1$ *for all* i,
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A *signed circuit* C is a circuit with signs $s_i \in \{+, -\}$ such that

$$\sum_{i=1,\dots,k} s_i m_i \alpha_{c_i} = 0 \in \mathbb{Z}^r$$

for some $m_i \in \mathbb{N}$.

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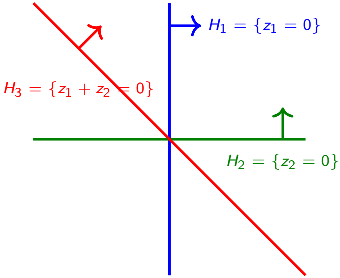

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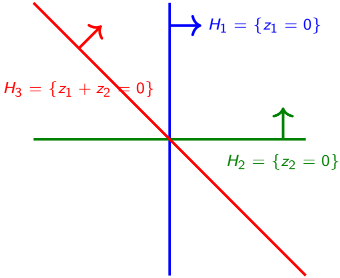

The circuit relations are $\omega_1^+ \omega_2^+ \omega_3^- = 0$ and $\omega_1^- \omega_2^- \omega_3^+ = 0$

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Theorem (Feichtner-Ziegler '00, de Longueville-Schultz '01)

If $b > 1$, then

$$H^*(M_{\mathcal{A}}^{0,b}; \mathbb{Z}) \simeq \mathbb{Z}[\omega_1, \omega_2, \ldots, \omega_n] \big/ (\partial \omega_C \mid C \text{ circuit}) + (\omega_i^2)_i$$

where the generators ω_i are in degree $b - 1$.

Unimodular arrangements

Assume $a > 0$.

Definition

An abelian arrangement is *unimodular* if for any $I \subseteq \mathcal{A}$ the intersection $\bigcap_{\alpha \in I} H_\alpha$ is connected or empty.

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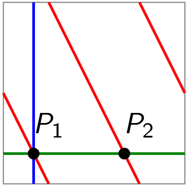

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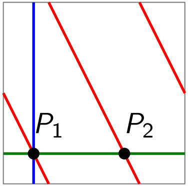

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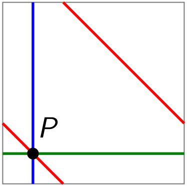

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The toric arrangement on the left is not unimodular, the right one is unimodular.

Toric arrangements

Consider the case of $(1,1)$-arrangements.
The cohomology of $\mathbb{C}^* \setminus \{1\}$ is generated by the forms $\omega = \frac{1}{2\pi\sqrt{-1}} \mathrm{d}\log(z-1)$ and $\psi = \frac{1}{2\pi\sqrt{-1}} \mathrm{d}\log z$ with relation $\omega\psi = 0$.

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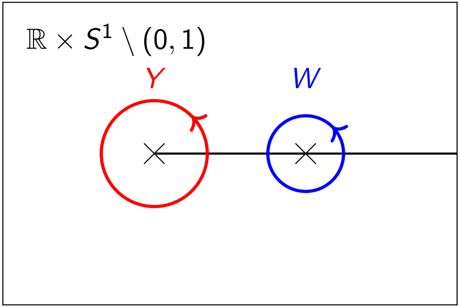

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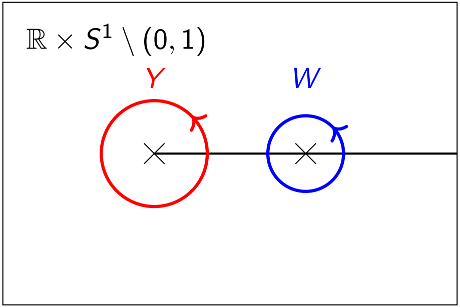

The cohomological basis $\{\omega, \psi\}$ is dual to the homological basis $\{W, Y\}$.

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Defines the elements

$$\omega_i = \frac{1}{2\pi\sqrt{-1}} \operatorname{d}\log(\alpha_i(\underline{z}) - 1) \text{ and } \psi_i = \frac{1}{2\pi\sqrt{-1}} \operatorname{d}\log \alpha_i(\underline{z}).$$

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Theorem (De Concini-Procesi '05)

The cohomology ring $H^*(M^{1,1}_{\mathcal{A}};\mathbb{C})$ of a unimodular toric arrangement is the ring $H^*((\mathbb{C}^*)^r)[\omega_i]_i$ with relations:

1. $\omega_i\psi_i = 0$ *for all* i,
2. $\partial\omega_C + l.o.t. = 0$ *for each signed circuit* $C = (c_1, c_2, \dots, c_k)$.

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Unimodularity implies that the cohomology is generated in degree one.

Let $m(I) = |\pi_0(\bigcap_{\alpha \in I} H_\alpha)|$ be the number of connected components (i.e. the multiplicity function of the *arithmetic matroid*).

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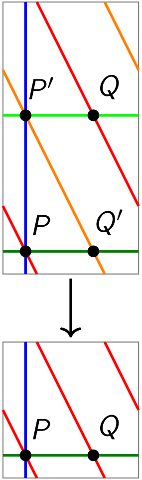
Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. '20)

The cohomology ring $H^*(M_{\mathcal{A}}^{1,1};\mathbb{Z})$ of a toric arrangement is the $H^*((\mathbb{C}^*)^r)$-algebra generated by $\omega_{W,I}$ with I independent set and $W \in \pi_0(\bigcap_{\alpha\in I} H_\alpha)$ with relations:

1. $\omega_{W,I}\psi_i = 0$ *for all* $I \ni i$,
2. $\omega_{W,I}\omega_{Z,J} = \pm \sum_{L\in\pi_0(W\cap Z)} \omega_{L,I\sqcup J}$
3. $\sum_{\text{some } A\subsetneq X} \pm \frac{m(A)}{m(A\cup B)}\omega_{W,A}\psi_B = 0,$ *for each generalized circuit* X *and each* $L \in \pi_0(\bigcap_{\alpha\in C} H_\alpha)$.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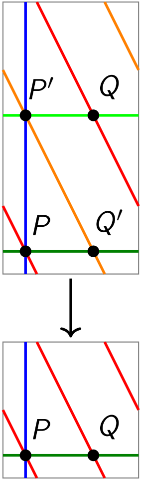

De Concini-Procesi result implies

$$\omega_1\omega_2 - \omega_1\omega_3 + \omega_2\omega_3 - \psi_2\omega_3 = 0,$$
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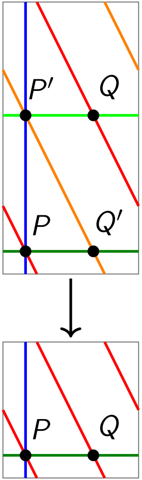

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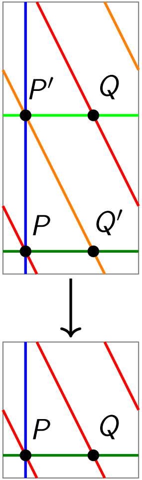
Setting $\omega_{P,23}$ such that $\pi^*\omega_{P,23} = \omega_2\omega_3 + \omega_2'\omega_3'$,



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Setting $\omega_{P,23}$ such that $\pi^*\omega_{P,23} = \omega_2\omega_3 + \omega_2'\omega_3'$, the CDDMP construction yields

$$\omega_1\omega_2 - \omega_1\omega_3 + \omega_{P,23} - \frac{1}{2}\psi_2\omega_3 = 0.$$

Abelian arrangements

We consider the general case of non-compact abelian arrangements, i.e. $b > 0$.

Theorem (Liu,Tran,Yoshinaga '21)

If $b > 0$, the homology $H_*(M_{\mathcal{A}}^{a,b}; \mathbb{Z})$ is torsion free and the Poincaré polynomial is

$$P_{M_{\mathcal{A}}^{a,b}}(t) = (-t^{a+b-1})^r \chi_{\mathcal{A}}^a \left(-\frac{(1+t)^a}{t^{a+b-1}} \right).$$

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2. $\omega_{W,I}\omega_{Z,J} = \pm \sum_{L \in \pi_0(W \cap Z)} \omega_{L, I \sqcup J}$
3. *for each generalized circuit* $X = C \sqcup F$ *and each* $L \in \pi_0(\bigcap_{\alpha \in X} H_\alpha)$
$$\sum_{B \subseteq C^+} \pm \frac{m(A)}{m(A \cup \widetilde{B})} \omega_{W,A} \psi_{\widetilde{B}} - \sum_{B \subseteq C^-} \pm \frac{m(A)}{m(A \cup \widetilde{B})} \omega_{W,A} \psi_{\widetilde{B}} = 0,$$
where $A = X \setminus B$ *and* $\widetilde{B} = B \setminus \min(B)$.

Sketch of proof

1. Consider the embedding $j\colon \mathbb{R} \setminus \{0\} \hookrightarrow \mathbb{R}^b \times (S^1)^a \setminus \{e\}$. Its *pushforward* in cohomology is $j_*(\omega^+) = \omega$, $j_*(\omega^-) = \psi - \omega$, and $j_*(1) = \psi$ (if $b > 1$ we set $\psi \equiv 0$).

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$$\begin{tikzcd} H^0(\mathbb{R}\setminus\{0\}) \arrow[r, "j_*"] \arrow[d, "\mathrm{PD}"'] & H^{a+b-1}(G\setminus\{e\}) \\ H_1^{\mathrm{BM}}(\mathbb{R}\setminus\{0\}) \arrow[r, "j_*^{\mathrm{BM}}"'] & H_1^{\mathrm{BM}}(G\setminus\{e\}) \arrow[u, "\mathrm{PD}^{-1}"'] \end{tikzcd}$$

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H^0(\mathbb{R} \setminus \{0\}) \arrow[r, "j_*"] \arrow[d, "\mathrm{PD}"'] & H^{a+b-1}(G \setminus \{e\}) \\
H_1^{\mathrm{BM}}(\mathbb{R} \setminus \{0\}) \arrow[r, "j_*^{\mathrm{BM}}"'] & H_1^{\mathrm{BM}}(G \setminus \{e\}) \arrow[u, "\mathrm{PD}^{-1}"']
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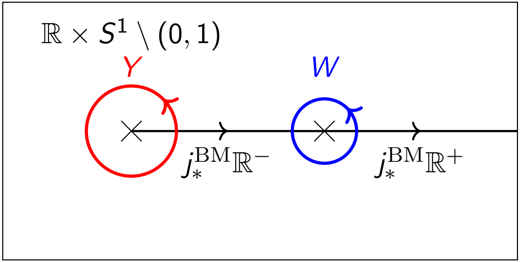

Sketch of proof

1. Consider the embedding $j \colon \mathbb{R} \setminus \{0\} \hookrightarrow \mathbb{R}^b \times (S^1)^a \setminus \{e\}$. Its *pushforward* in cohomology is $j_*(\omega^+) = \omega$, $j_*(\omega^-) = \psi - \omega$, and $j_*(1) = \psi$ (if $b > 1$ we set $\psi \equiv 0$).

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2. Consider a unimodular arrangement $\mathcal{A}$ and use Künneth formula for $j\colon M_{\mathcal{A}}^{0,1} \to M_{\mathcal{A}}^{a,b}$:
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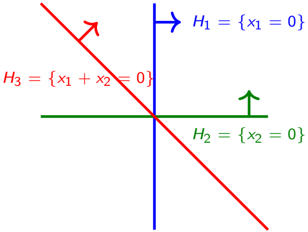
5. Use a variation of the Briskorn lemma and deletion-contraction argument.

Example



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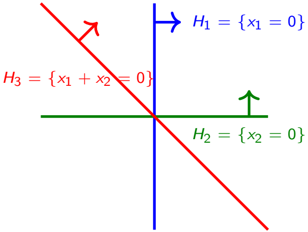

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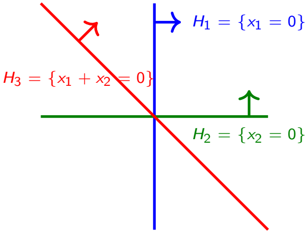

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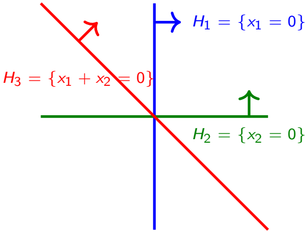

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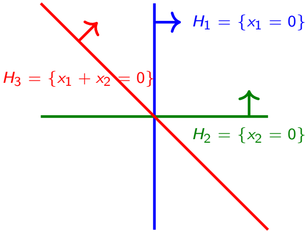

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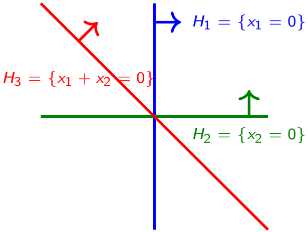

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In particular, $H^*(\mathrm{Conf}_n(\mathbb{R} \times (S^1)^a)) \not\simeq \mathrm{gr}_{\mathcal{F}} H^*(\mathrm{Conf}_n(\mathbb{R} \times (S^1)^a))$ as rings.

Other applications

1. Formality of abelian arrangements. It is already known in the cases $(0, b)$ [Brieskorn '73, Feichtner, Yuzvinsky '05] and $(1, 1)$ [De Concini, Procesi '05, Dupont '16]

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3. Explicit basis for Borel-Moore homology of abelian arrangements.

Thanks for listening!

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