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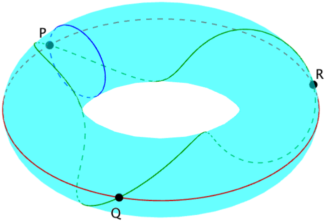
Wonderful models for toric arrangements

Arrangements at Home: Combinatorial Aspects



May 2020

Talk is being recorded.

Covered topics:

Arrangements of subtori

Wonderful model

Cohomology

Divisorial case

Let $T \cong (\mathbb{C}^*)^n$ be an algebraic torus. A *layer* is a coset for a subtorus:

$$W_{\Gamma,\phi} := \{t \in T \mid \chi(t) = \phi(\chi) \quad \forall \chi \in \Gamma\}$$

for some $\Gamma < \mathrm{Hom}(T, \mathbb{C}^*) = X^*(T) \cong \mathbb{Z}^n$ and $\phi \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$.
For $W = W_{\Gamma,\phi}$, define $\Gamma_W = \Gamma$.

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An *arrangement of subtori* $\mathcal{A}$ is a finite collection of layers. We are interesting in computing the cohomology of the *complement* $M(\mathcal{A}) = T \setminus \bigcup \mathcal{A}$.

We call a *toric arrangement* an arrangement of subtori of codimension one.

A *wonderful model* for the complement $M(\mathcal{A})$ is a smooth projective variety X containing $M(\mathcal{A})$ as open set whose complement $D = X \setminus M(\mathcal{A})$ is a *simple normal crossing divisor*, ie the irreducible components of D are smooth and "intersect locally as coordinate hyperplanes" in $\mathbb{C}^d$.

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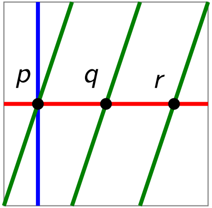
Theorem (Moci '12)

There exists a smooth variety $X_{\mathcal{G}}$ (non-compact) containing $M(\mathcal{A})$ whose complement is a simple normal crossing divisor.

Idea: consider $M(\mathcal{A}) \subset T$ and blow up some layers in $T \setminus M(\mathcal{A})$.

The *poset of layers* is the set $\mathcal{S}(\mathcal{A})$ of all connected components of intersections ordered by reverse inclusion.

Example



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$$t_1^3 t_2^{-1} = 1$$

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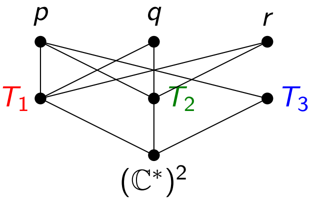

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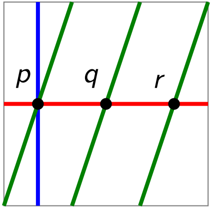

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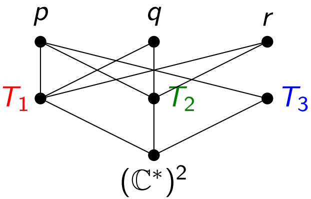

A *building set* $\mathcal{G} \subset \mathcal{S}(\mathcal{A})$ has the property that for all $W \in \mathcal{S}(\mathcal{A})$ the maximal elements $G_1, \ldots, G_k$ in $\mathcal{G}_{\leq W}$ satisfy:

1. $W \in \vee_{i=1}^k G_1$,
2. $\operatorname{codim} W = \sum_{i=1}^k \operatorname{codim} G_i$.

These G_is are called $\mathcal{G}$*-factors* of W.

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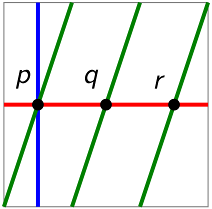

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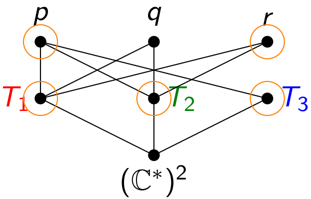

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Let $\mathcal{G} \subset \mathcal{S}(\mathcal{A})$, let $X_{\mathcal{G}}$ be the (iterated) blow up in T of the elements in $\mathcal{G}$.

Proposition

If $\mathcal{G}$ is a building set, then $X_{\mathcal{G}} \setminus M(\mathcal{A})$ is a simple normal crossing divisor.

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Definition

A *$\mathcal{G}$-nested set* is $\mathcal{T} \subset \mathcal{G}$ such that:

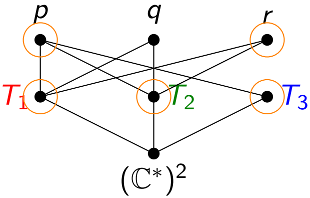
1. the minimal elements $T_1, \dots, T_k$ of $\mathcal{T}$ are the $\mathcal{G}$-factors of some W,
2. the sets $\mathcal{T}_{>T_i}$ are $\mathcal{G}$-nested for $i = 1, \dots, k$.

Example

If $\mathcal{G} = \mathcal{S}(\mathcal{A}) \setminus \{T\}$ is the maximal building set, the $\mathcal{G}$-nested set are the chains in $\mathcal{S}(\mathcal{A}) \setminus \{T\}$.

Example

The $\mathcal{G}$-nested sets are $\{T_1, T_2\}, \{p, T_1\}, \{p, T_2\}, \{p, T_3\}, \{r, T_1\}$, $\{r, T_2\}$.



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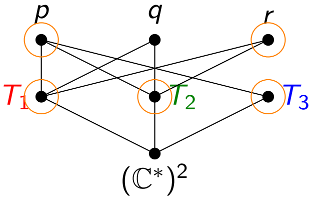

Proposition

$\mathcal{T} \subseteq \mathcal{G}$ is $\mathcal{G}$-nested set if and only if the corresponding divisors in $X_{\mathcal{G}}$ intersect.

Idea [De Concini, Gaiffi]: consider a toric variety X_Δ and the inclusions $M(\mathcal{A}) \subset T \subset X_\Delta$.
Problem: the closure of a layer $W \in \mathcal{A}$ may be singular and the intersection with the divisors at "infinity" (i.e. in $X_\Delta \setminus T$) may not be transversal.

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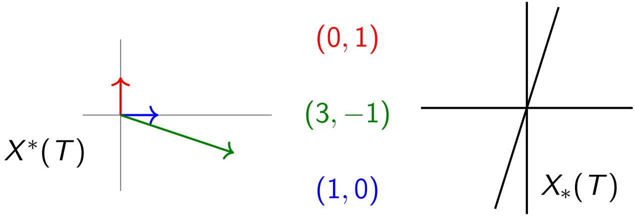
Definition

A fan $\Delta \subset X_*(T)$ is *equal sign* w.r.t. W if there exists a basis $\chi_1, \ldots, \chi_k$ of Γ_W such that every cone of Δ is entirely contained in H_i^+ or in H_i^-. A fan $\Delta \subset X_*(T)$ is *equal sign* w.r.t. $\mathcal{A} = \{W_1, \ldots, W_r\}$ if it is equal sign w.r.t. W_i for all i.



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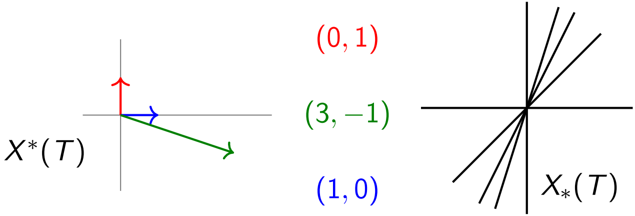

Lemma (De Concini, Gaiffi)

If Δ is a smooth complete fan with the equal sign property w.r.t. $\mathcal{A}$ then $\overline{W} \subset X_\Delta$ is smooth and intersects transversally the divisors at "infinity".

Let $X_{\Delta,\mathcal{G}}$ be the (iterated) blow up of X_Δ along the closure of layers in $\mathcal{G}$.

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Let $X_{\Delta,\mathcal{G}}$ be the (iterated) blow up of X_Δ along the closure of layers in $\mathcal{G}$.

Theorem (De Concini, Gaiffi '19)

If Δ is a smooth complete fan with the equal sign property w.r.t. $\mathcal{A}$ and $\mathcal{G} \subset \mathcal{S}(\mathcal{A})$ is a building set, then $X_{\Delta,\mathcal{G}}$ is a wonderful model for $M(\mathcal{A})$.

Example

With our previous choice, $X_{\Delta,\mathcal{G}}$ is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in 8 points.

Theorem (Keel '92)

Let $i\colon Y \subset X$ such that $i^*\colon H^\bullet(X) \twoheadrightarrow H^\bullet(Y)$ and call I the kernel. Then

$$H^\bullet(\mathrm{Bl}_Y(X)) \cong \frac{H^\bullet(X)[t]}{(It, P(t))},$$

where $i^*(P(t)) = c(N_Y(X), t)$ is any lifting of the Chern polynomial of the normal bundle.

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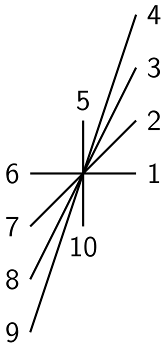
Theorem

The cohomology of the toric variety X_Δ is the algebra $\mathbb{Q}[c_r]_{r \in R_\Delta}$ with relations:

1. $c_{r_1} c_{r_2} \dots c_{r_k}$ *if the rays $r_1, r_2, \dots, r_k$ do not span a cone of Δ,*
2. $\sum_{r \in R_\Delta} \langle \chi, r \rangle c_r$ *for any $\chi \in X^*(T)$.*

Example



Relations 1 give: $c_1c_3 = c_1c_4 = c_1c_5 = c_1c_6 = c_1c_7 = c_1c_8 = c_1c_9 = \ldots = c_8c_{10} = 0$, relations 2 give:

$c_1 + c_2 + c_3 + c_4 - c_6 - c_7 - c_8 - c_9 = 0$,
$c_2 + 2c_3 + 3c_4 + c_5 - c_7 - 2c_8 - 3c_9 - c_{10} = 0$.

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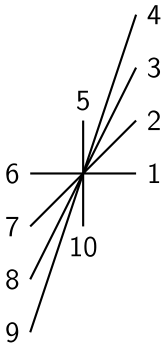

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The cohomology of $\mathrm{Bl}_p(X_\Delta)$ is

$$H^*(X_\Delta)[t] \Big/ \big(tc_i, t^2 - (c_1 + c_2 + c_3 + c_4)(c_2 + 2c_3 + 3c_4 + c_5)\big).$$

The irreducible components of $X_{\Delta,\mathcal{G}} \setminus M(\mathcal{A})$ are $\overline{W}$ for $W \in \mathcal{G}$ and C_r for $r \in R_\Delta$. The divisors $\overline{W}$, C_r for $W \in A \subset \mathcal{G}$ and $r \in B \subset R_\Delta$ have nonempty intersection if and only if;

1. A is $\mathcal{G}$-nested,
2. B spans a cone of Δ,
3. for all $W \in A$, $r \in B$ we have $r \in \operatorname{Ann}\Gamma_W$.

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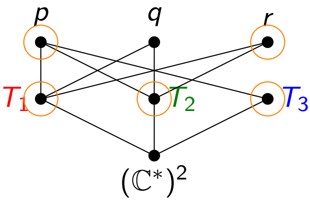

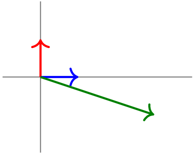

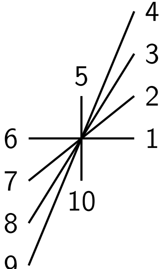

$C_1 \cap C_2 \neq \emptyset,\ T_1 \cap T_2 \neq \emptyset,\ T_1 \cap C_6 \neq \emptyset,$
$C_1 \cap C_6 = T_1 \cap T_3 = T_1 \cap C_2 = \emptyset.$

Theorem (De Concini, Gaiffi '19)

The cohomology of the intersection of divisors associated with A and B is the algebra generated by t_W, c_r, $W \in \mathcal{G}$, $r \in R_\Delta$, with relations:

- T1 $\prod_{r \in B'} c_r$ *if* $B \cup B'$ *do not span a cone of* Δ,
- T2 $\sum_{r \in R_\Delta} \langle \chi, r \rangle c_r$ *for any* $\chi \in X^*(T)$,
- W1 c_r *if* $r \notin \operatorname{Ann} \Gamma_W$ *for all* $W \in A$,
- W2 $c_r t_W$ *if* $r \notin \operatorname{Ann} \Gamma_W$,
- W3a *for all* $W \in \mathcal{G}$ *and all* $A' \subseteq \mathcal{G}_{<W}$, *the relations*

$$P_W^V\Big(\sum_{L \in \mathcal{G}_{\geq W}} -t_L \Big) \prod_{L \in A'} t_L,$$

  where $V \in \vee_{L \in A_{<W} \cup A'} L$ *and* $V < W$,
- W3b $\prod_{W \in A'} t_W$ *if* $A' \cup A$ *is not* $\mathcal{G}$*-nested or* $B \not\subset \bigcap_{W \in A'} \operatorname{Ann} \Gamma_W$.

The *Morgan algebra* M of a simple normal crossing divisor $D = \cup_{i=1}^{n} D_i \subset X$ is the differential graded algebra

$$\bigoplus_{A \subset [n]} H^{\bullet}(\cap_{i \in A} D_i)$$

with product induced by the cup product

$$H^p(\cap_{i \in A} D_i) \otimes H^q(\cap_{i \in B} D_i) \to H^{p+q}(\cap_{i \in A \sqcup B} D_i)$$

and differential induced by the Gysin maps

$$H^p(\cap_{i \in A} D_i) \to H^{p+2}(\cap_{i \in A \setminus \{a\}} D_i).$$

Theorem (Morgan '78)

The dga M *is a model for* $X \setminus D$*, hence* $H(\mathrm{M}, \mathrm{d}) = H(X \setminus D)$.

Theorem (Moci, P. '20)

The Morgan algebra $\mathrm{M}(\mathcal{A}, \Delta, \mathcal{G})$ *for the wonderful model* $X_{\Delta,\mathcal{G}}$ *is generated by* s_W, t_W, b_r, c_r *for* $W \in \mathcal{G}$ *and* $r \in R_\Delta$ *with relations:*

1. *if* $r \notin \operatorname{Ann}\Gamma_W$: $s_W b_r$, $s_W c_r$, $t_W b_r$, *and* $t_W c_r$,
2. $\prod_{W\in A} s_W \prod_{W \in B} t_W$ *if* $A \cup B$ *is not a* $\mathcal{G}$*-nested set,*
3. $\prod_{r\in A} b_r \prod_{r\in B} c_r$ *if* $A \cup B$ *is not a cone in* Δ,
4. $\sum_{r\in R_\Delta} \langle \chi, r\rangle c_r$ *for every* $\chi \in \Lambda$,
5. $P_W^V(\sum_{L\in\mathcal{G}_{\geq W}} -t_L) \prod_{L\in A} s_L \prod_{L\in B} t_L$ *for* $W \in \mathcal{G}$, *and* $A, B \subseteq \mathcal{G}_{<W}$, *where* $V \in \vee_{L\in A\cup B} L$ *and* $V < W$.

and differential d *defined on generators by* $\mathrm{d}(s_W) = t_W$, $\mathrm{d}(b_j) = c_j$, *and* $\mathrm{d}(t_W) = \mathrm{d}(c_j) = 0$.

Now we assume that $\mathcal{A} = \{H_1, \dots, H_n\}$ contains only codimension 1 subtori and let $\mathcal{G} = \mathcal{S}(\mathcal{A}) \setminus \{T\}$ be the maximal building set.
In this case $\Gamma_{H_i} \cong \mathbb{Z}$ is generated by one element $\pm\chi_i$.
The *rank function* $A \mapsto \operatorname{codim}(\cap_{a \in A} H_a)$ define a *matroid* $([n], \mathrm{rk})$.

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Definition (Branden-Moci '14, D'Adderio-Moci '13)

The *multiplicity function* $A \mapsto \#c.c. \cap_{a \in A} H_a$ defines an *arithmetic matroid* $([n], \mathrm{rk}, m)$.

The choice of one between χ_i and $-\chi_i$ defines an *oriented matroid*: for any minimal linear relation $\sum_{i \in C} n_i \chi_i = 0$ the *orientation* of C is $i \mapsto \operatorname{sgn}(n_i)$.

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Definition (P. '20)

The rank function, multiplicity function, and orientation defines a *oriented arithmetic matroid*.

If $W = \vee_{a \in A} H_a$ is *complete intersection*, a lift of the Chern polynomial $P_W(t)$ is the following:

$$P_W(t) = \prod_{a \in A} \Big(t - \sum_{r \in R_\Delta} \min(0, \langle \chi_a, r \rangle) c_r \Big).$$

Another possible choice is

$$P_W(t) = t^{|A|} + \prod_{a \in A} \Big(\sum_{r \in R_\Delta} - \min(0, \langle \chi_a, r \rangle) c_r \Big).$$

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Example

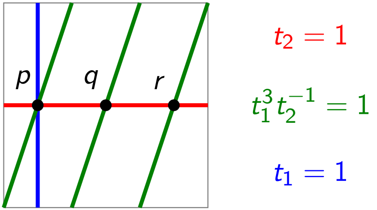
In $\ker \mathrm{d} \subset \mathrm{M}(\mathcal{A}, \Delta, \mathcal{G})$ contains $\psi_i := \sum_{r\in\mathcal{R}_\Delta} \langle \chi_i, r\rangle b_j$ and $\omega_{H_i,i} := -\sum_{W>H_i} s_W - \sum_{r\in R_\Delta} \min(0, \langle \chi_i, r\rangle) b_r$ by relations 4 and 5.

Theorem (Moci, P. '20)

There exists a subalgebra of $\mathrm{M}(\mathcal{A}, \Delta, \mathcal{G})$ *isomorphic to* $H^\bullet(T)[\omega_{W,A}]/I$ *(A independent,* $W \in \vee_{a\in A} H_a$*) where* I *is generated by:*

1. $\omega_{W,A}\omega_{W',A'} = \pm \sum_{L\in W\vee W'} \omega_{L,A\sqcup A'}$,
2. $\omega_{W,A}\psi = 0$ *if* $\psi_{|W} = 0$,
3. *for* $X \subset [n]$ *such that* $\mathrm{rk}(X) = |X| - 1$ *(*$X = C \sqcup F$*) and associated relation* $\sum_{i\in C} n_i\chi_i$*, for* $L \in \vee_{x\in X} H_x$*:*
$$\sum_{\substack{F\subseteq A\subsetneq X \\ C\setminus A \text{ positroids}}} (-1)^{|X_{<j}|} \frac{m(A)}{m(X\setminus\{j\})} \omega_{W,A}\psi_B = 0$$
where $j = \min(C\setminus A)$*,* $B = C\setminus(A\cup\{j\})$ *and* $\psi_B = \prod_{b\in B}\psi_b$ *an element in* $H^\bullet(T)$*.*

and is isomorphic to $H(M(\mathcal{A}); \mathbb{Q})$*.*



The linear relation is $\chi_1 + \chi_2 - 3\chi_3 = 0$

$$\omega_1\omega_2 = \omega_{p,12} + \omega_{q,12} + \omega_{r,12} \qquad \omega_1\psi_2 = 0$$

$$\omega_1\omega_3 = \omega_{p,13} \qquad \omega_2(3\psi_1 - \psi_2) = 0$$

$$\omega_2\omega_3 = \omega_{p,23} \qquad \omega_3\psi_1 = 0$$

$$\omega_{p,12} - \omega_{p,13} + \omega_{p,23} + \omega_3\psi_2 = 0$$

Theorem (Callegaro, D'Adderio, Delucchi, Migliorini, P. '19)

There exists a subalgebra of the algebraic de Rham complex of $M(\mathcal{A})$ isomorphic to $H^\bullet(T)[\overline{\omega}_{W,A}]/I$ (A independent, $W \in \vee_{a\in A} H_a$) where I is generated by:

1. $\overline{\omega}_{W,A}\overline{\omega}_{W',A'} = \pm \sum_{L\in W\vee W'} \overline{\omega}_{L,A\sqcup A'}$,
2. $\overline{\omega}_{W,A}\psi = 0$ *if* $\psi_{|W} = 0$,
3. *for* $X \subset [n]$ *such that* $\mathrm{rk}(X) = |X| - 1$ ($X = C \sqcup F$) *and associated relation* $\sum_{i\in C} n_i\chi_i$, *for* $L \in \vee_{x\in X} H_x$:

$$\sum_{j\in C} \sum_{\substack{F\subseteq A\subseteq X\setminus\{j\} \\ |B| \text{ even}}} (-1)^{|X_{<j}|} \frac{m(A)}{m(X\setminus\{j\})} \overline{\omega}_{W,A} c_B \psi_B = 0$$

where $B = C \setminus (A \cup \{j\})$, $c_B = \prod_{b\in B} \mathrm{sgn}(n_b)$, *and* $\psi_B = \prod_{b\in B} \psi_b$ *an element in* $H^\bullet(T)$,

and it is isomorphic to $H(M(\mathcal{A}); \mathbb{Q})$.

	Subspace	Toric
Additive co-homology	Goresky-MacPhearson '80	De Concini-Procesi '05 [1], Deshpande '18, Moci-P. '20
Wonderful model	De Concini-Procesi '95	De Concini-Gaiffi '19
Small model	Yuzvinsky '89	Moci-P. '20 [1]
Cohomology ring	Orlik-Solomon '80 [1], Feichtner-Ziegler '00, de Longueville-Schultz '00 Deligne-Goresky-MacPhearson '00	C.D.D.M.P. '19 [1], Moci-P. '20 [1]
Non-realizable	Adiprasito-Huh-Katz '18, Bibby-Denham-Feichtner '19	???

[1] Divisorial case

Thanks for listening!

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