Adaptive tangential interpolation in rational Krylov subspaces for MIMO model reduction

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joint work with Vladimir Druskin and Mikhail Zaslavsky (Schlumberger-Doll Research)

Model Order Reduction

Given the continuous time-invariant linear system

$$\mathbf{x}'(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \qquad \mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right), \ A \in \mathbb{C}^{n\times n}$$
$$\mathbf{y}(t) = C\mathbf{x}(t), \quad \mathbf{x}(0) = x_0$$

and $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{s\times n}$

Analyse the construction of a reduced system

$$\hat{\mathbf{\Sigma}} = \left(\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \end{array}\right)$$

with $\tilde{A}$ of size $m \ll n$

Projection methods and Linear Dynamical Systems

Time-invariant linear system:

$$\mathbf{x}'(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \qquad \mathbf{x}(0) = x_0$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

Emphasis: A large dimensions, $W(A) \subset \mathbb{C}^-$

Projection methods: the general idea

Given space $K \subset \mathbb{R}^n$ of size m and (orthonormal) basis V_m,

$$A \to A_m = V_m^* A V_m, \quad B \to B_m = V_m^* B, \quad C \to C_m = C V_m$$

Reduced problem uses: A_m, B_m, C_m

Typical applications

Approximation of the transfer function:

$$\mathcal{H}(\omega) = C(A-\omega I)^{-1}B, \quad \omega \in i\mathbb{R} \quad \approx \quad \mathcal{H}_m(\omega) := C_m(A_m-\omega I)^{-1}B_m$$

The Lyapunov matrix equation: $\quad AX + XA^* + BB^* = 0$

Galerkin approximation by projection: $\quad V_m$ orth. basis,
$K = \mathrm{range}(V_m)$

$$X \approx X_m = V_m Y V_m^*, \quad R_m := AX_m + X_m A^* + BB^*$$

$$\text{with} \qquad R_m \perp K \quad \Leftrightarrow \quad V_m^* R_m V_m = 0$$

that is,

$$V_m^* A V_m Y + Y V_m^* A^* V_m + V_m^* B B^* V_m = 0 \qquad \text{Small size}$$

Projection methods: the general idea

Choices of K in the literature:

- Standard Krylov subspace:
  $K_m(A, B) = \mathrm{range}([B, AB, \ldots, A^{m-1}B])$

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- Shift-Invert Krylov subspace:
  $K_m((A-\sigma I)^{-1}, B) = \mathrm{range}([B, (A-\sigma I)^{-1}B, \ldots, (A-\sigma I)^{-(m-1)}B])$;
  often $\sigma = 0$

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- Extended Krylov subspace:
  $\mathbf{EK}_m(A, B) = K_m(A, B) + K_m(A^{-1}, A^{-1}B)$
- Rational Krylov subspace:
  $K_m(A, B, \mathbf{s}) = \mathrm{range}([(A - s_1 I)^{-1}B, (A - s_2 I)^{-1}B, \ldots, (A - s_m I)^{-1}B]$
  usually $\mathbf{s} = [s_1, \ldots, s_m]$ a-priori

Rational Krylov Subspaces. A long tradition...

$$K_m(A, B, \mathbf{s}) = \mathrm{range}([(A - s_1 I)^{-1} B, (A - s_2 I)^{-1} B, \ldots, (A - s_m I)^{-1} B])$$

- Eigenvalue problems (Ruhe, 1984)
- Model Order Reduction (transfer function evaluation)
- ADI for linear matrix equations

Rational Krylov Subspaces in MOR. Choice of poles.

$$K_m(A, B, \mathbf{s}) = \mathrm{range}([(A - s_1 I)^{-1}B, (A - s_2 I)^{-1}B, \ldots, (A - s_m I)^{-1}B])$$

cf. General discussion in Antoulas, 2005.

Various attempts:

- Gallivan, Grimme, Van Dooren (1996–, ad-hoc poles)
- Penzl (1999-2000, ADI shifts - preprocessing, Ritz values)
- ....
- Sabino (2006 - tuning within preprocessing)

- IRKA – Gugercin, Antoulas, Beattie (2008)

Adaptive choice of poles for RKS. $B = b \in \mathbb{C}^n$

$$K_m(A, b, \mathbf{s}) = \mathrm{range}([(A - s_1 I)^{-1} b, (A - s_2 I)^{-1} b, \ldots, (A - s_m I)^{-1} b])$$

$\mathbf{s} = [s_1, \ldots, s_m]$ to be chosen sequentially

The fundamental idea: Assume you wish to solve

$$(A - sI)x = b$$

with a Galerkin procedure in $K_m(A, b, \mathbf{s})$. Let V_m be orth. basis. The residual satisfies:

$$b - (A - sI)x_m = \frac{r_m(A)b}{r_m(s)}, \qquad r_m(z) = \prod_{j=1}^{m} \frac{z - \lambda_j}{z - s_j}$$

with $\lambda_j = \mathrm{eigs}(V_m^* A V_m)$. Moreover,

$$\|r_m(A)b\| = \min_{\theta_1, \ldots, \theta_m} \| \prod_{j=1}^{m} (A - \theta_j I)(A - s_j I)^{-1} b\|$$

Adaptive choice of poles for RKS. Cont'd

$$r_m(z) = \prod_{j=1}^{m} \frac{z-\lambda_j}{z-s_j}, \qquad \lambda_j = \mathrm{eigs}(V_m^* A V_m)$$

For A symmetric:

$$s_{m+1} := \arg\left(\max_{s\in[-\lambda_{\max}, -\lambda_{\min}]} \frac{1}{|r_m(s)|} \right)$$

$[\lambda_{\min}, \lambda_{\max}] \approx \mathrm{spec}(A)$ (Druskin, Lieberman, Zaslavski (SISC 2010))

For A nonsymmetric:

$$s_{m+1} := \arg\left(\max_{s\in\partial\mathcal{S}_m} \frac{1}{|r_m(s)|} \right)$$

where $\mathcal{S}_m \subset \mathbb{C}^+$ approximately encloses the eigenvalues of $-A$

(Druskin, Simoncini (S&C Lett. 2011))

Motivated by potential theory arguments...

The multiple input case. $B \in \mathbb{C}^{n \times p}$, $p \gg 1$

Straightforward generalization:

$$K_m(A, B, \mathbf{s}) = \mathrm{range}([(A - s_1 I)^{-1} B, (A - s_2 I)^{-1} B, \ldots, (A - s_m I)^{-1} B])$$

$\Uparrow$ Easy to implement

$\Downarrow$ Generates possibly redundant information

$\Downarrow$ Memory/Computational costs inefficient

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An alternative:

$$\mathbf{T}_m = \mathrm{range}([(A - s_1 I)^{-1}Bd_1, \ldots, (A - s_m I)^{-1}Bd_m])$$

with an adaptive choice of (s_i, d_i)

Tangential rational Krylov subspace

$$\mathbf{T}_m = \text{range}([(A - s_1 I)^{-1} B d_1, \ldots, (A - s_m I)^{-1} B d_m]) = range(V_m)$$

Some properties:

- $\mathcal{H}(s_i) d_i = \mathcal{H}_m(s_i) d_i, \quad i = 1, \ldots, m$
- For A symmetric,

$$d_i^* \frac{d}{ds} \left. \mathcal{H}(s) \right|_{s=s_i} d_i = d_i^* \frac{d}{ds} \left. \mathcal{H}_m(s) \right|_{s=s_i} d_i, \quad i = 1, \ldots, m$$

- If $v_{m+1} = (A - s_{m+1} I)^{-1} B d_{m+1}$ and
  $R_m(s) = (A - sI) V_m (H_m - sI)^{-1} V_m^* B - B$, then

$$\begin{aligned} \mathbf{T}_{m+1} &:= \text{range}([V_m, v_{m+1}]) \\ &= \text{range}([V_m, (A - s_{m+1} I)^{-1} R_m(s_{m+1}) d_{m+1}]), \end{aligned}$$

and $\dim(\mathbf{T}_{m+1})) = m + 1$ if and only if $R_m(s_{m+1}) d_{m+1} \neq 0$

Adaptive choice of poles and directions.

$$\mathbf{T}_m = \mathrm{range}([(A - s_1 I)^{-1} B d_1, \ldots, (A - s_m I)^{-1} B d_m])$$

Single direction:

$$(d_{m+1}, s_{m+1}) = \arg \max_{\substack{s \in \mathcal{S}_m \\ d \in \mathbb{R}^p, \|d\|=1}} \|R_m(s) d\|$$

In fact:

1. Compute s_{m+1} where $\|R_m(s)\|$ is largest
2. Compute d_{m+1} as principal SVD direction of $R_m(s_{m+1})$

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Multiple directions:

2'. Compute $d_{m+1} \in \mathbb{R}^{p \times \ell}$ as ℓ principal SVD directions of $R_m(s_{m+1})$

$$\ell \quad s.t. \quad \sigma^{(i)} > \frac{1}{10} \sigma^{(1)}, \quad i = 1, \ldots, \ell$$

where $\sigma^{(k)}$, $k = 1, \ldots, p$ are the sing.values of $R_m(s_{m+1})$

Related characterizations

Optimal $\mathbf{H}_2$ model reduction: IRKA

Determines projection spaces so that

$$\mathcal{H}_m(\omega) = C_m(A_m - \omega I)^{-1} B_m$$

satisfies first order necessary conditions for optimal $\mathbf{H}_2$ reduction:

$$\mathcal{H}(-\lambda_i) = \mathcal{H}_m(-\lambda_i), \quad \mathcal{H}'(-\lambda_i) = \mathcal{H}'_m(-\lambda_i)$$

where λ_i's are the simple poles of $\mathcal{H}_m$ (Gugercin Antoulas Beattie, '08)

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Tangential interpolation in the MIMO case:

If B, C^T have multiple columns, then first order necessary conditions:

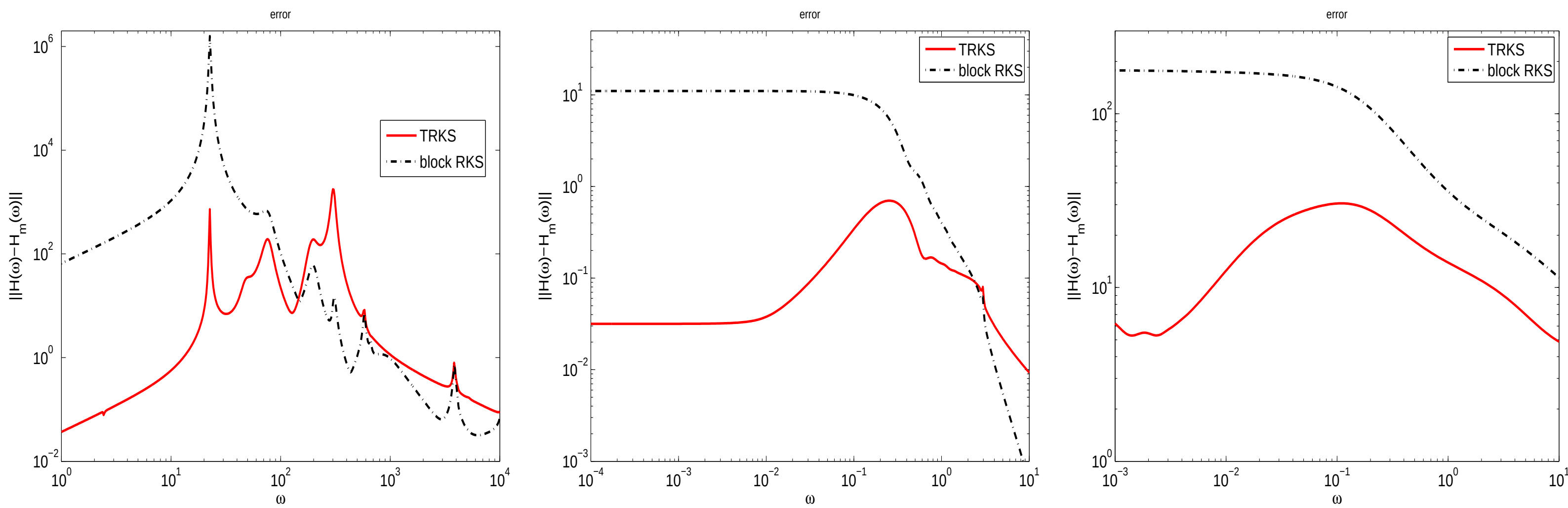
$$\mathcal{H}(-\lambda_i)x_i = \mathcal{H}_m(-\lambda_i)x_i \quad \mathcal{H}'(-\lambda_i)x_i = \mathcal{H}'_m(-\lambda_i)x_i$$

x_i are projected left eigenvectors of $\mathcal{H}_m$ (also for right eigenvec's)

(Benner,Bunse-Gerstner,Kubalinska,Vossen,Wilczek,Van Dooren,Gallivan,Absil,...)

Some numerical examples. Transfer function approximation

$$||\mathcal{H}(\omega) - \mathcal{H}_m(\omega)||, \quad \omega \in i[\alpha, \beta]$$



Data from Oberwolfach collection: CD Player, EADY, FLOW

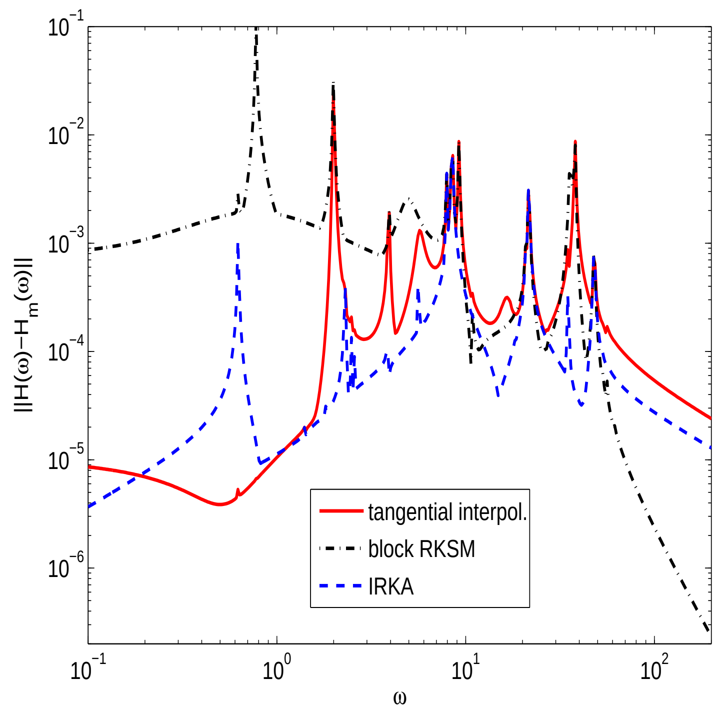
Original block RKSM vs Tangential approach (TRKS)

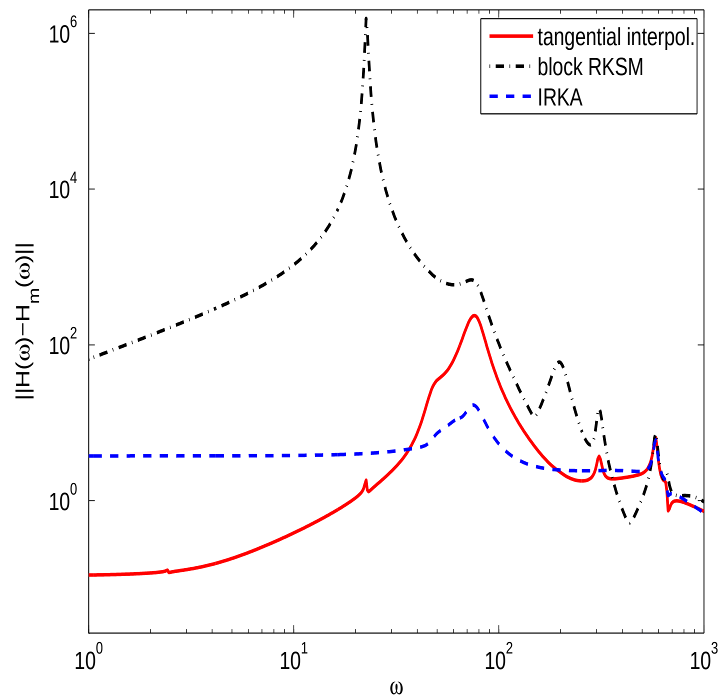
Final space dimension = 10 ($p = 2, 10, 5$ in the three cases, resp.)

Real poles

Comparisons with optimal IRKA

$$\|\mathcal{H}(\omega) - \mathcal{H}_m(\omega)\|, \qquad \omega \in i[\alpha, \beta]$$





Data from Oberwolfach collection: ISS, CD Player

Original block RKSM vs Tangential approach vs IRKA

Final space dimension $= 10$ ($p = 3, 2$ in the two cases, resp.)

Complex poles

Some numerical examples. Lyapunov equation

Tangential approximation space

$$\mathrm{range}([B, (A - s_1 I)^{-1} B d_1, \ldots, (A - s_m I)^{-1} B d_m]), \quad d_i \in \mathbb{R}^{p \times \ell_i}$$

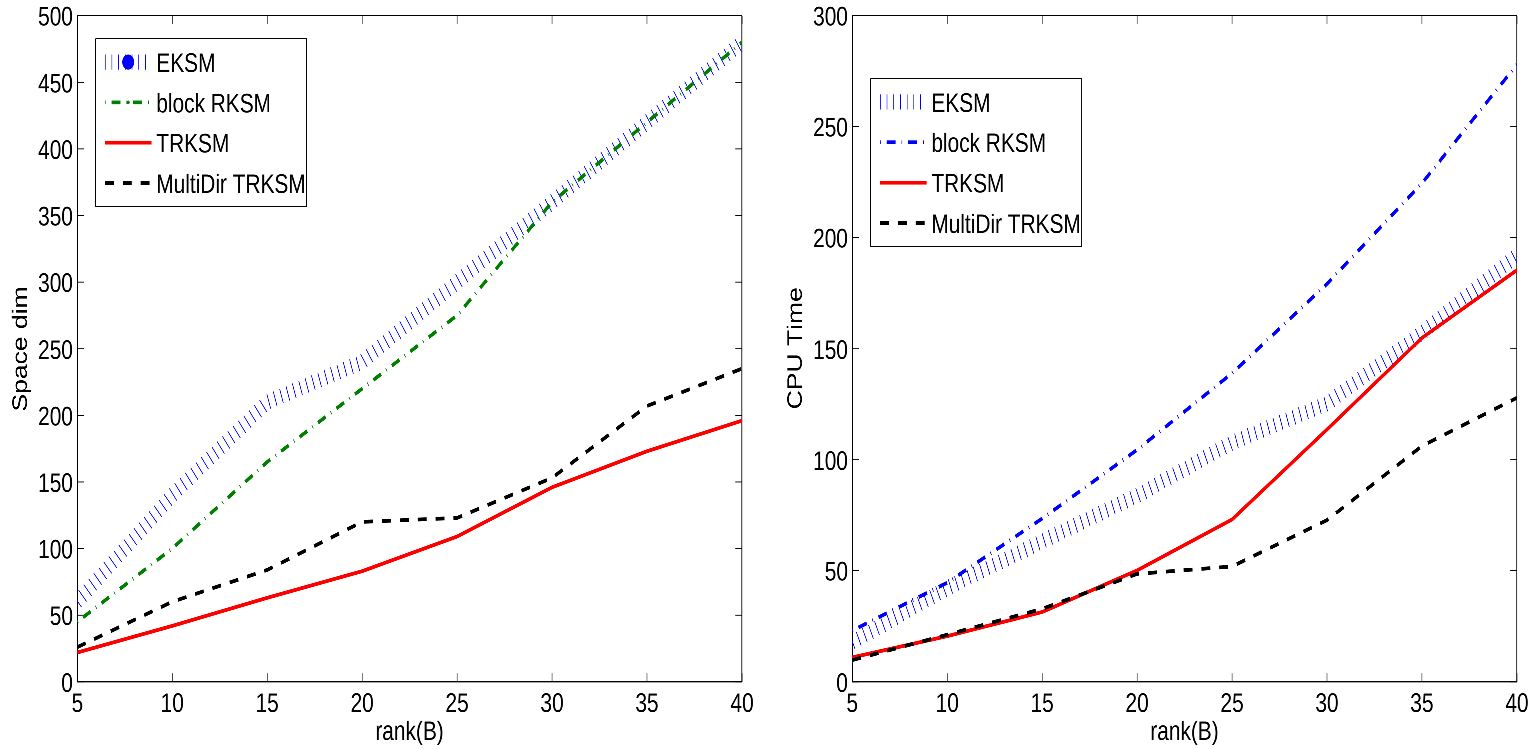
Computational considerations

- Cheap evaluation of the residual norm
- Adaptive selection of poles and directions at cost indep. of problem size

Some numerical examples. Lyapunov equation

Pb size: $90,000$ (FD discr.: $\mathcal{L}(u) = (e^{-xy}u_x)_x + (-e^{xy}u_y)_y$ in $[0,1]^2$)

B random

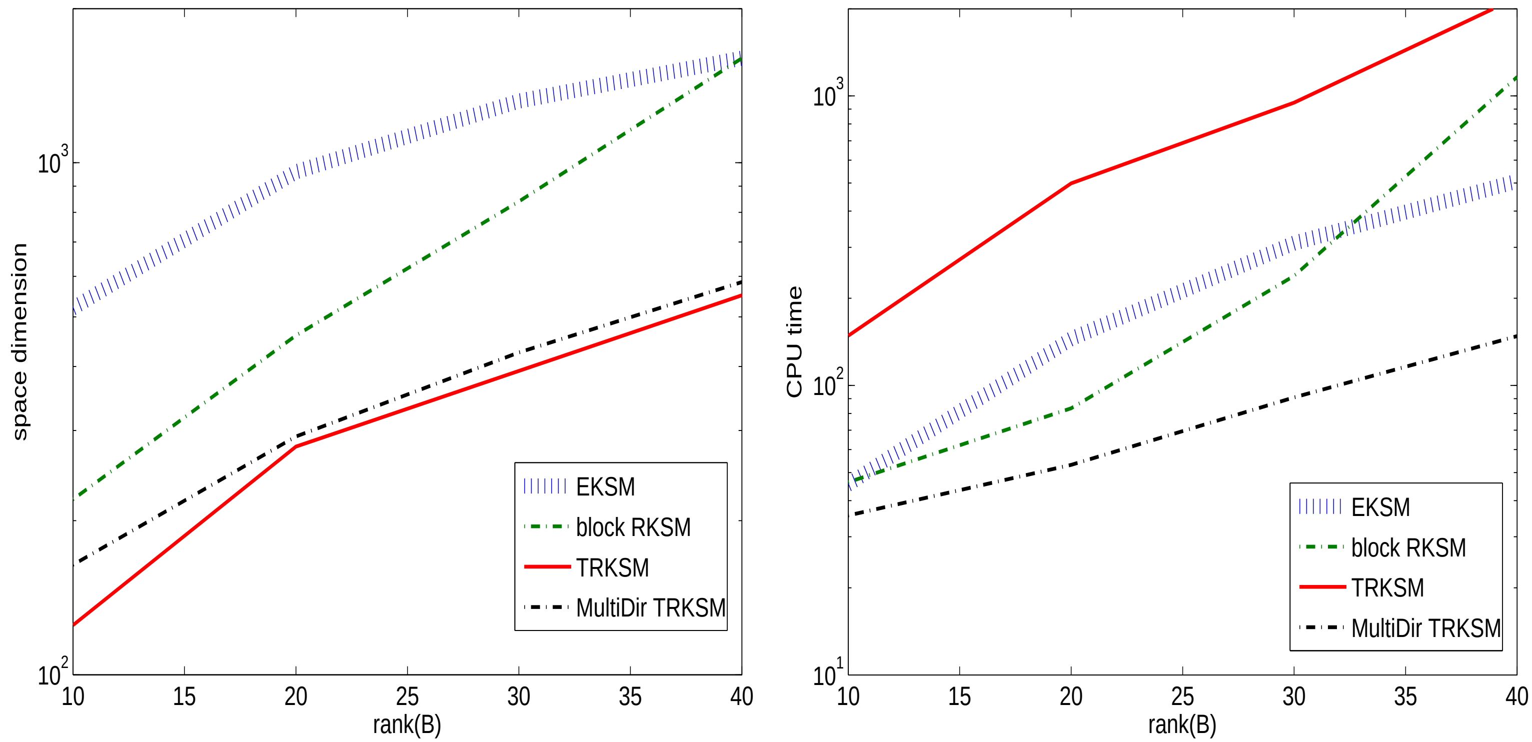


Inner solves: PCG and AMG preconditioning

Also Extended Krylov subspace method (EKSM) included

Some numerical examples. Lyapunov equation

matrix CHIP (Oberwolfach), size $20,090$, B random



Inner solves: direct method

Also Extended Krylov subspace method (EKSM) included

Conclusions

- Tangential approach valuable device for MIMO systems
- Idea possibly useful also for standard mrhs linear systems

Tech.rep.:

Adaptive tangential interpolation in rational Krylov subspaces for MIMO model reduction,

V. Druskin, V. Simoncini and M. Zaslavsky, Nov. 2012.

available at: `www.dm.unibo.it/~simoncin`

Survey paper: *Computational methods for linear matrix equations*,

V.Simoncini, March. 2013.