Università di Bologna

Advances in numerical projection methods for MOR of large-scale linear dynamical systems

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Model Order Reduction

Given the continuous-time system

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right), \quad A \in \mathbb{C}^{n\times n}$$

Analyse the construction of a reduced system

$$\hat{\mathbf{\Sigma}} = \left(\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \end{array}\right)$$

with $\tilde{A}$ of size $m \ll n$, and issues associated with its accuracy.

Applications: signal processing, system and control theory

Projection methods and Linear Dynamical Systems

- Solvers for the Lyapunov matrix equation
- Approximation of the matrix Transfer function
- Approximation of Hankel singular values by balanced truncation

Solving the Lyapunov equation. The problem

Approximate soln X to:

$$AX + XA^\top + BB^\top = 0$$

$A \in \mathbb{R}^{n\times n}$ positive real $\qquad B \in \mathbb{R}^{n\times s}, \qquad s \geq 1$

Solving the Lyapunov equation. The problem

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Time-invariant linear system:

$$\mathbf{x}'(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \qquad \mathbf{x}(0) = x_0$$

Analytic solution:

$$X = \int_0^\infty e^{-tA} BB^\top e^{-tA^\top} dt = \int_0^\infty xx^\top dt \quad \text{with} \;\; x = \exp(-tA)B$$

see, e.g., Antoulas '05, Benner '06

Standard Krylov subspace projection

$$X \approx X_m \qquad X_m \in \mathcal{K}$$

Galerkin condition: $R := AX_m + X_m A^\top + bb^\top \quad \perp \quad \mathcal{K}$

$$V_m^\top R V_m = 0 \qquad\qquad \mathcal{K} = \mathrm{range}(V_m)$$

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$$V_m^\top R V_m = 0 \qquad\qquad \mathcal{K} = \mathrm{range}(V_m)$$

Assume $V_m^\top V_m = I_m$ and let $X_m := V_m Y_m V_m^\top$.

Projected Lyapunov equation:

$$(V_m^\top A V_m) Y_m + Y_m (V_m^\top A^\top V_m) \quad + \quad V_m^\top bb^\top V_m = 0$$

$$\Updownarrow$$

$$T_m Y_m + Y_m T_m^\top \quad + \quad e_1 e_1^\top = 0$$

with $b = V_m e_1$ (Saad, '90, for $\mathcal{K} = \mathcal{K}_m(A, b) = \mathrm{span}\{b, Ab, \ldots, A^{m-1}b\}$)

Standard Krylov projection. In quest of error bounds

$$AX + XA^\top + BB^\top = 0, \qquad X \approx X_m \in \mathcal{K}_m(A, B)$$

$$\|X - X_m\| \ \leq \ ??$$

Standard Krylov projection. In quest of error bounds

$$AX + XA^\top + BB^\top = 0, \qquad X \approx X_m \in \mathcal{K}_m(A, B)$$

$$\|X - X_m\| \le ??$$

(Simoncini & Druskin '09). Analytic solution:

$$X = \int_0^\infty e^{-tA} BB^\top e^{-tA^\top} dt = \int_0^\infty xx^\top dt$$

with $x = \exp(-tA)B$, $\qquad B = b,\ \|b\| = 1$

Let $\alpha_{\min} = \lambda_{\min}((A + A^\top)/2) > 0$. Then

$$\|x\| \le \exp(-t\alpha_{\min})\|B\|$$

First (key) step

Krylov subspace proj.: $X_m = V_m Y_m V_m^\top$, range$(V_m) = \mathcal{K}_m(A, b)$

$$T_m Y_m + Y_m T_m^\top + e_1 e_1^\top = 0$$

Clearly,

$$X_m = V_m \left(\int_0^\infty e^{-tT_m} e_1 e_1^\top e^{-tT_m^\top} dt \right) V_m^\top = \int_0^\infty x_m x_m^\top dt$$

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II step: $\|X - X_m\| = \| \int_0^\infty (xx^\top - x_m x_m^\top) dt \|$, so that

$$\|X - X_m\| \le \int_0^\infty \|xx^\top - x_m x_m^\top\| dt \le 2 \int_0^\infty e^{-t\alpha_{\min}} \|x - x_m\| dt$$

The case of A symmetric

A symmetric $\Rightarrow \;\; \alpha_{\min} = \lambda_{\min}(A)$

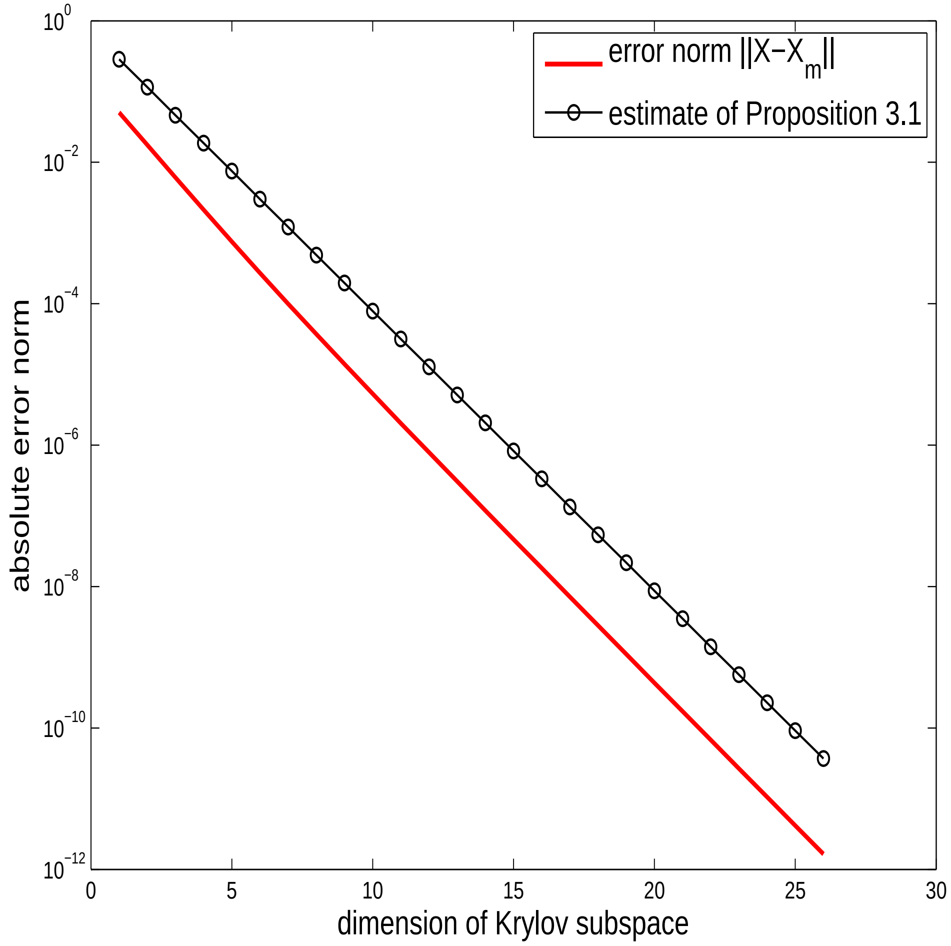
Let $0 < \hat{\lambda}_{\min} \leq \ldots \leq \hat{\lambda}_{\max}$ eigs of $A + \lambda_{\min} I$, $\qquad \hat{\kappa} := \frac{\hat{\lambda}_{\max}}{\hat{\lambda}_{\min}}$

Then

$$\|X - X_m\| \;\; \leq \;\; \frac{\sqrt{\hat{\kappa}} + 1}{\hat{\lambda}_{\min}\sqrt{\hat{\kappa}}} \left(\frac{\sqrt{\hat{\kappa}} - 1}{\sqrt{\hat{\kappa}} + 1} \right)^m$$

Note: same rate as CG for $(A + \lambda_{\min} I)z = b$

The case of A symmetric. An example



A: 400×400 diagonal with uniformly distributed eigenvalues in $[1, 10]$

$(\alpha_{\min} = \lambda_{\min} = 1)$

The case of $W(A)$ in an ellipse

Assume $W(A) \subseteq E \subset \mathbb{C}^+$

$\big($$E$ ellipse of center $(c, 0)$, foci $(c \pm d, 0)$ and major semi-axis $a$$\big)$

Then

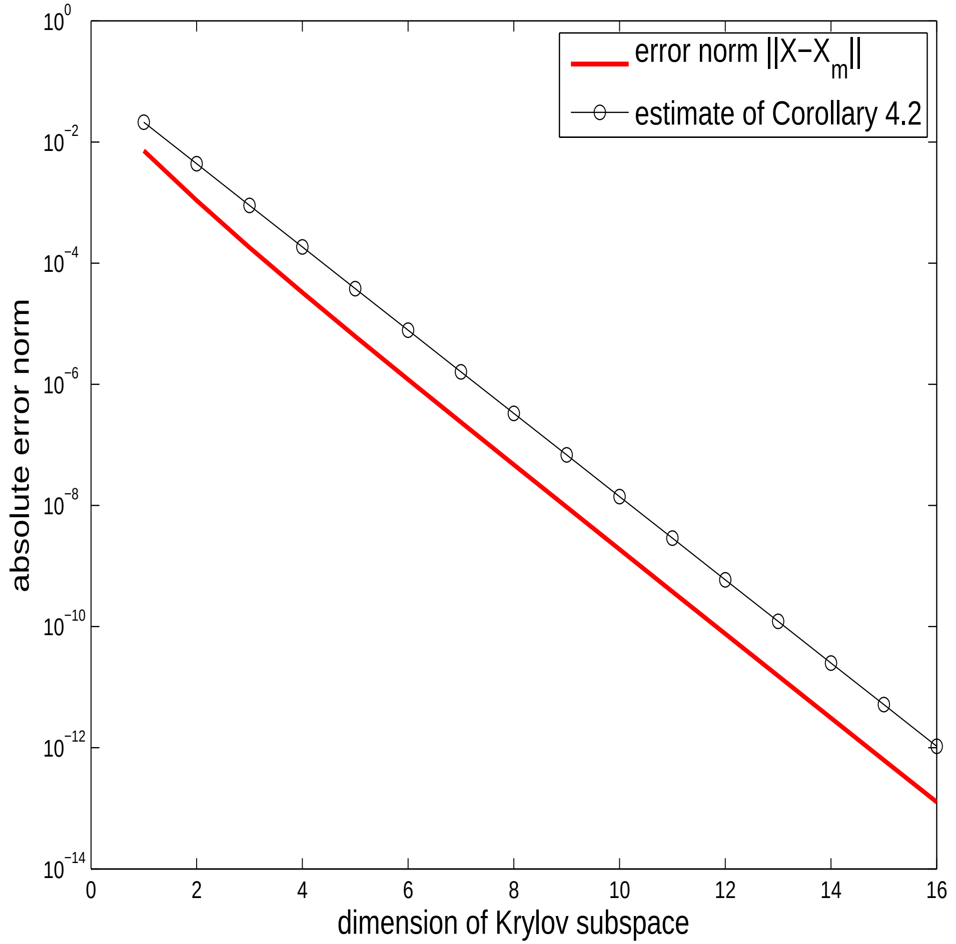
$$\|X - X_m\| \leq \frac{4}{\alpha_{\min}} \frac{r_2}{r_2 - r} \left(\frac{r}{r_2}\right)^m$$

where

$$r = \frac{a}{d} + \sqrt{\left(\frac{a}{d}\right)^2 - 1}, \quad r_2 = \frac{c + \alpha_{\min}}{d} + \sqrt{\left(\frac{c + \alpha_{\min}}{d}\right)^2 - 1}$$

Note: same rate as FOM for $(A + \alpha_{\min} I)z = b$

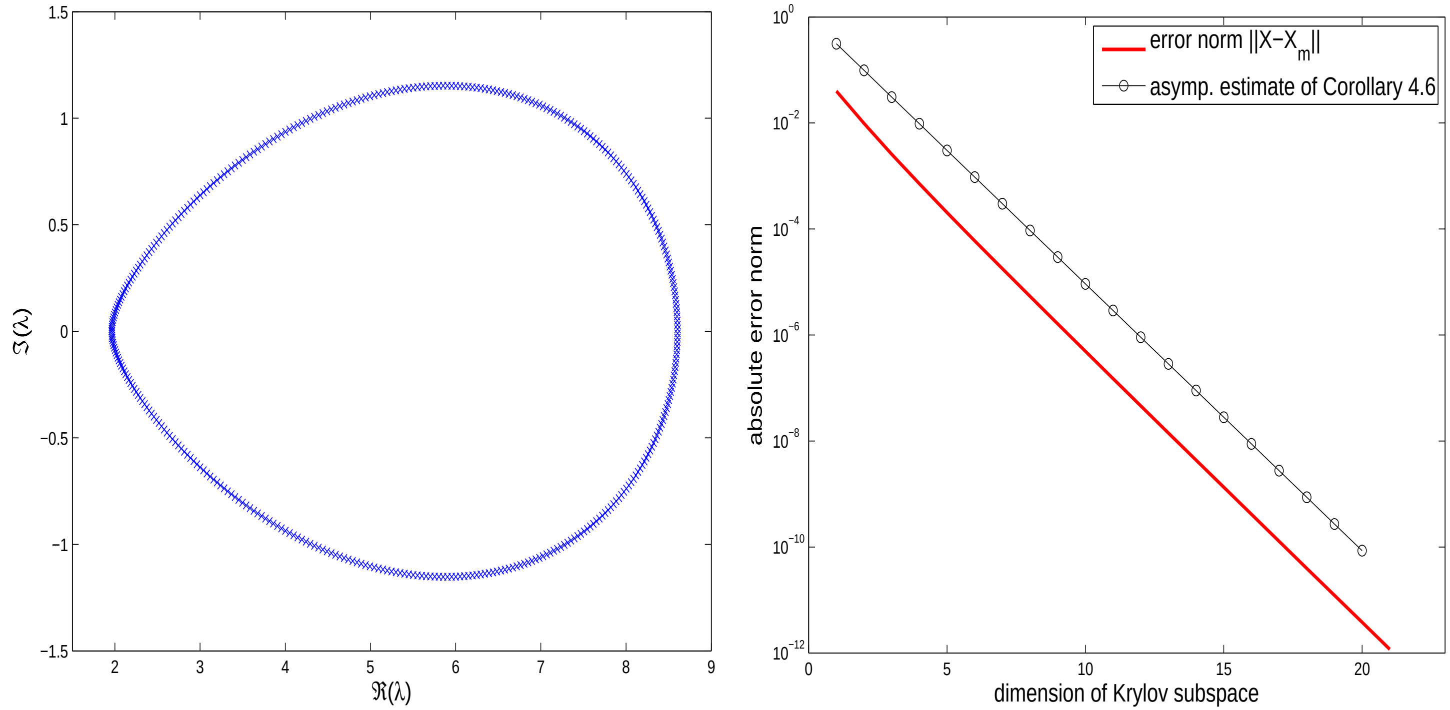
The case of $W(A)$ in an ellipse.



A normal with eigenvalues on an elliptic curve

The case of $W(A)$ in a wedge-shaped set. An example

Generalization to a wedge-shaped convex set of $\mathbb{C}^+$.



A: diagonal (normal) matrix on the wedge-shaped curve.

Cyclic low rank Smith method

(ADI made efficient)

(see, e.g., Li 2000, Penzl 2000)

$$\begin{aligned} X_0 = 0,\ X_j &= -2p_j(A+p_jI)^{-1}BB^\top(A+p_jI)^{-\top} \quad j=1,\ldots,\ell \\ &\quad +(A+p_jI)^{-1}(A-p_jI)X_{j-1}(A-p_jI)^\top(A+p_jI)^{-\top} \end{aligned}$$

with

$$r_\ell(t) = \Pi_{j=1}^{\ell}(t-p_j), \quad \{p_1,\ldots,p_\ell\} = \operatorname{argmin} \max_{t\in\Lambda(A)} \left|\frac{r_\ell(t)}{r_\ell(-t)}\right|$$

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Convergence considerations:

Convergence depends on choice of $\{p_j\}$. For A spd:

$$\|X - X_\ell\| \approx \left(\frac{\sqrt{\kappa_{adi}}-2}{\sqrt{\kappa_{adi}}+2}\right)^\ell, \qquad \kappa_{adi} = \frac{\lambda_{\max}}{\lambda_{\min}}$$

Extended Krylov subspace method

Galerkin condition: $X_m \in \mathcal{K}$ s.t.

$$R := AX_m + X_m A^\top + bb^\top \quad \perp \quad \mathcal{K}$$

$$\mathcal{K} = \mathcal{K}_m(A, B) + \mathcal{K}_m(A^{-1}, A^{-1}B), \qquad \text{range}(\mathcal{V}_m) = \mathcal{K}$$

(Druskin & Knizhnerman '98, Simoncini '07) $\qquad X_m = \mathcal{V}_m Y_m \mathcal{V}_m^\top$

Projected Lyapunov equation:

$$(\mathcal{V}_m^\top A \mathcal{V}_m) Y_m + Y_m (\mathcal{V}_m^\top A^\top \mathcal{V}_m) \quad + \quad \mathcal{V}_m^\top b b^\top \mathcal{V}_m = 0$$

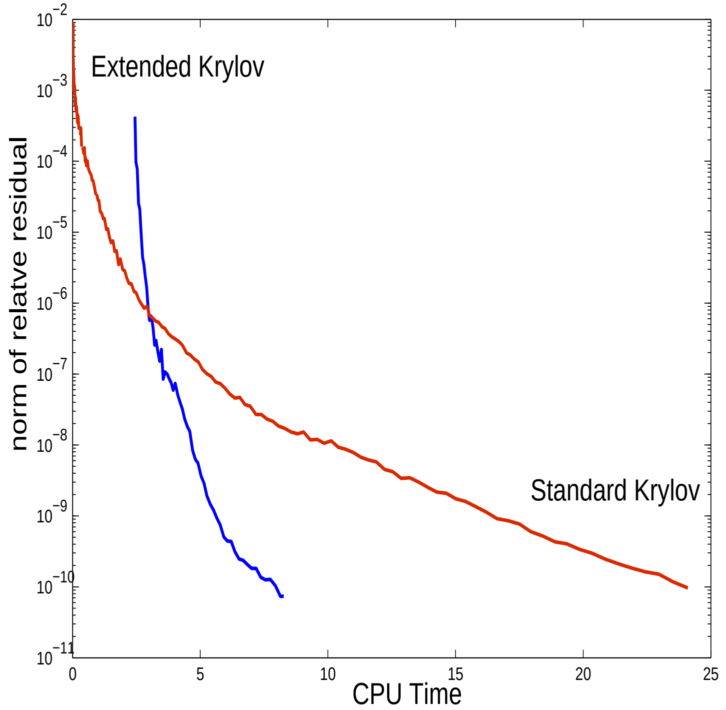
$$\Updownarrow$$

$$\mathcal{T}_m Y_m + Y_m \mathcal{T}_m^\top \quad + \quad e_1 e_1^\top = 0$$

Performance evaluation. I

$$\mathbf{x}' = \mathbf{x}_{xx} + \mathbf{x}_{yy} + \mathbf{x}_{zz} - 10x\mathbf{x}_x - 1000y\mathbf{x}_y - 10z\mathbf{x}_z + \mathbf{b}(x,y)\mathbf{u}(t)$$

A matrix $\quad 18^3 \times 18^3$



approximation space dim.: 146 (Standard Krylov) 112 (Extended Krylov)

Performance evaluation. II

Stopping criterion: norm of difference in solution

	s	EKSM		CF-ADI	
		time(#its)	dim.space	time (#its)	dim.space
Example	1	5.95 (12)	24	31.66 (6)	120
`rail_5177`	2	8.08 (10)	40	30.83 (5)	200
`tol`$=10^{-5}$	4	11.11 (7)	56	40.20 (5)	400
	7	18.12 (6)	84	54.22 (5)	700
Example (*)	1	38.95 (34)	68	588.68 (5)	150
`tol`$=10^{-8}$	2	50.50 (33)	132	633.41 (5)	300
	4	90.69 (33)	264	722.92 (5)	600
	7	204.91 (32)	448	857.57 (5)	1050

$$\mathbf{x}' = \mathbf{x}_{xx} + \mathbf{x}_{yy} + \mathbf{x}_{zz} - 10x\mathbf{x}_x - 1000y\mathbf{x}_y - 10z\mathbf{x}_z + \mathbf{b}(x,y)\mathbf{u}(t) \qquad (*)$$

Convergence analysis of Extended Krylov

General considerations

$$AX + XA^\top + BB^\top = 0$$

$$A^{-1}X + XA^{-\top} + A^{-1}BB^\top A^{-\top} = 0$$

Summing up for any $\gamma \in \mathbb{R}$, we obtain yet a Lyapunov equation:

$$(A+\gamma A^{-1})X + X(A^\top + \gamma A^{-\top}) + [B, \sqrt{\gamma}A^{-1}B][B^\top; \sqrt{\gamma}B^\top A^{-\top}] = 0$$

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$$(A+\gamma A^{-1})X + X(A^\top + \gamma A^{-\top}) + [B, \sqrt{\gamma}A^{-1}B][B^\top; \sqrt{\gamma}B^\top A^{-\top}] = 0$$

with $\mathcal{K}_m(A+\gamma A^{-1}, [B, \sqrt{\gamma}A^{-1}B]) \subsetneq \mathcal{K}_m(A, B) + \mathcal{K}_m(A^{-1}, A^{-1}B)$

$$\mathcal{A}X + X\mathcal{A}^\top + \mathcal{B}\mathcal{B}^\top = 0$$

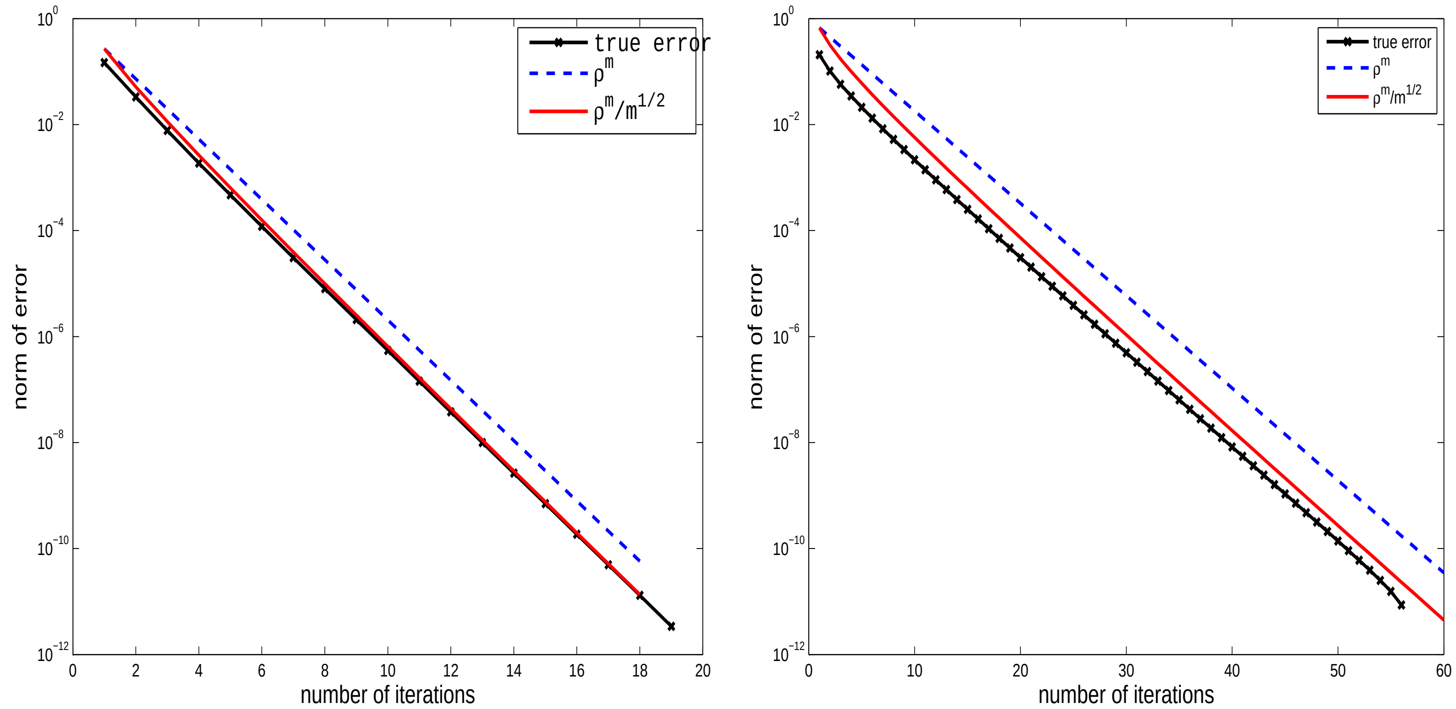
Convergence analysis of Extended Krylov: A symmetric pos.def.

Kressner & Tobler tr'09:

$$\|X - X_m\| \quad \lesssim \quad \Bigg(\underbrace{\left(\frac{\sqrt[4]{\kappa} - 1}{\sqrt[4]{\kappa} - 1} \right)^{\frac{1}{2}}}_{\rho} \Bigg)^{m} \qquad \kappa = \frac{\lambda_{\max}}{\lambda_{\min}}$$

A symmetric. An example

True error norm and asymptotic estimates for A symmetric.



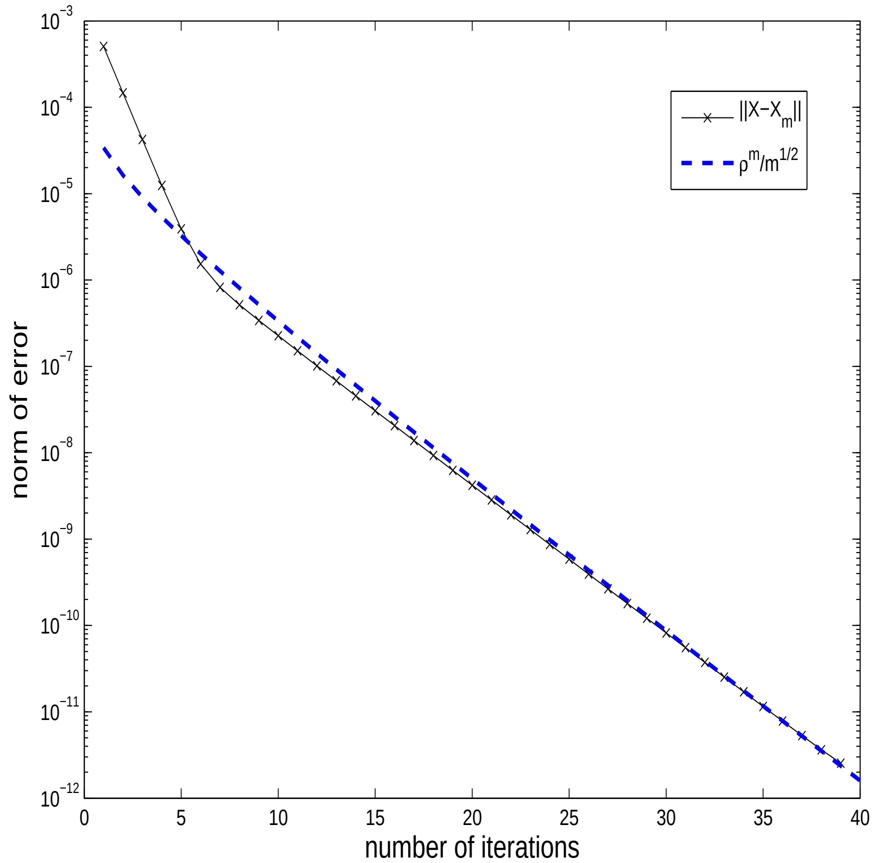
Left: Spectrum in $[0.1, 10]$. Right: Spectrum in $[0.01, 100]$

Knizhnerman & Simoncini (in prep.)

Convergence analysis of Extended Krylov: A nonsymmetric

$W(A) \subset \mathbb{C}^+$ a disk of center c and radius r.

$$\|X - X_m\| \lesssim \frac{1}{\sqrt{m}} \left(\frac{r^2}{4c^2 - 3r^2} \right)^m$$

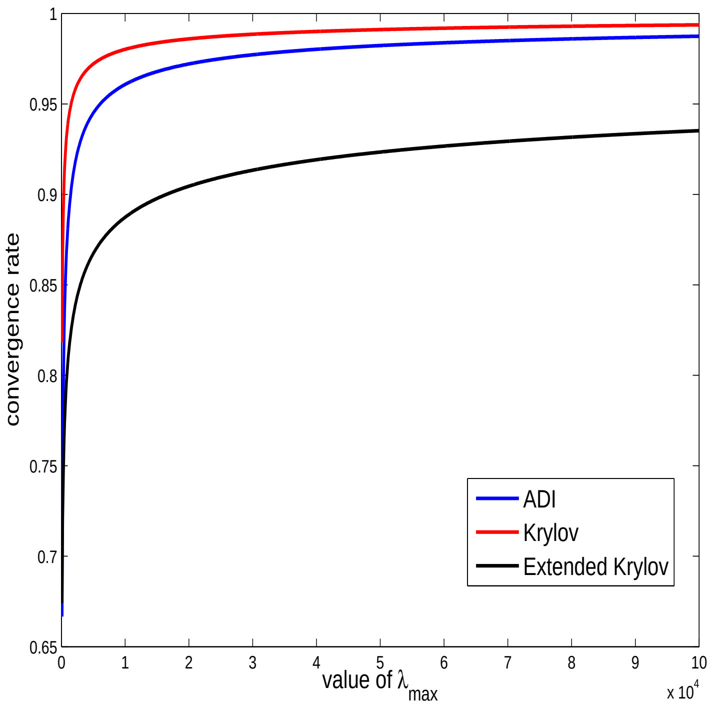


Knizhnerman & Simoncini (in prep.)

Comparison of convergence rates: A symmetric

ADI iteration: $\varepsilon_{adi,j} \approx \left(\frac{\sqrt{\kappa_{adi}}-2}{\sqrt{\kappa_{adi}}+2}\right)^j$ Standard Krylov: $\varepsilon_{kr,j} \approx \left(\frac{\sqrt{\hat{\kappa}}-1}{\sqrt{\hat{\kappa}}-1}\right)^j$

Extended Krylov: $\varepsilon_{ek,\ell} \approx \left(\left(\frac{\sqrt[4]{\kappa}-1}{\sqrt[4]{\kappa}-1}\right)^{1/2}\right)^\ell$



$\lambda_{\min} = 1, \ \lambda_{\max} \in [10^2, 10^5]$

Transfer function approximation (cf. MOR)

$$h(\sigma) = c^*(A - i\sigma I)^{-1}b, \quad \sigma \in [\alpha, \beta]$$

Given space $\mathcal{K}$ and V s.t. $\mathcal{K}$=range(V),

$$h(\sigma) \approx (V^*c)^*(V^*AV - \sigma I)^{-1}(V^*b)$$

For $\mathcal{K} = \mathcal{K}_m(A, b)$ (standard Krylov):

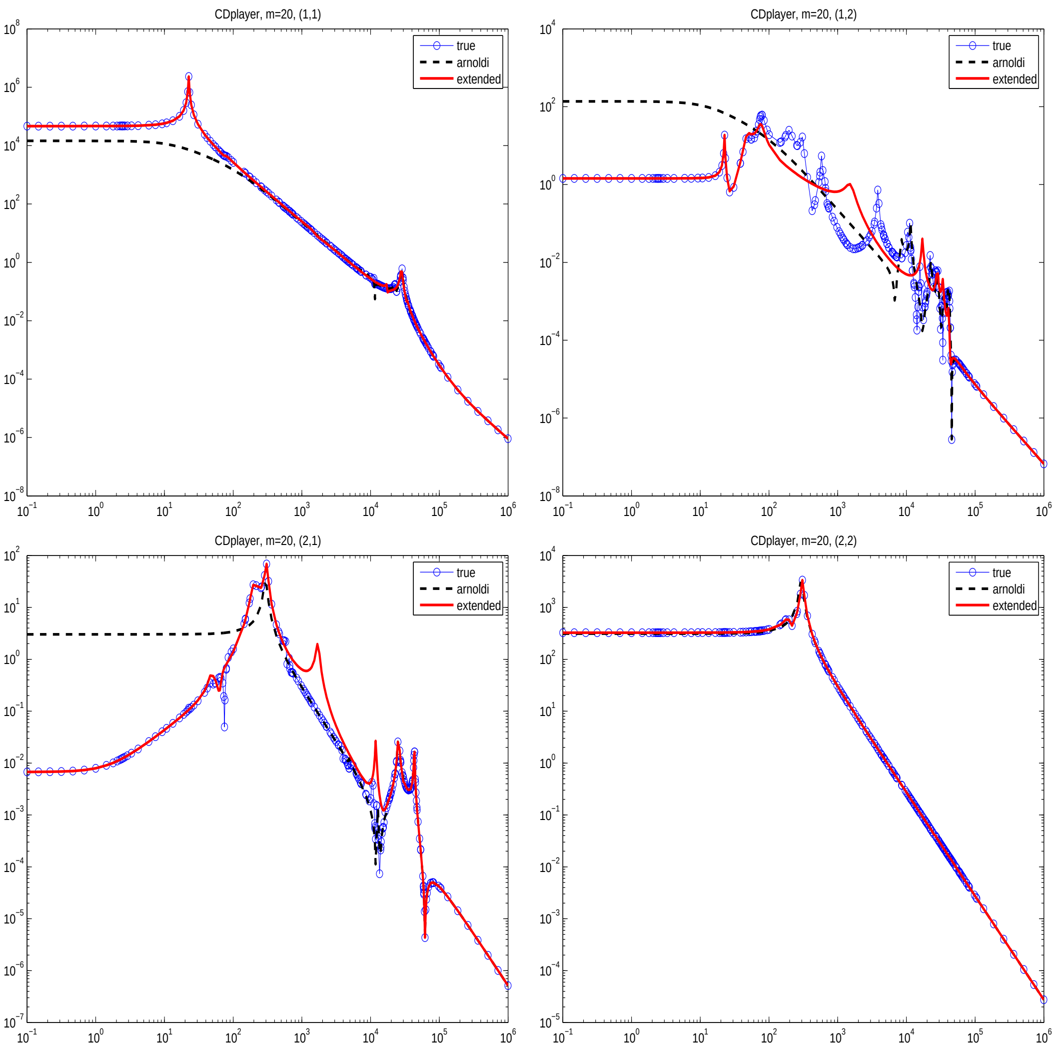
$$h_m(\sigma) = (V_m^*c)^*(H_m - \sigma I)^{-1}e_1\|b\|$$

For $\mathcal{K} = \mathcal{K}_m(A, b) + \mathcal{K}_m(A^{-1}, A^{-1}b)$ (EKSM):

$$h_m(\sigma) = (\mathcal{U}_m^*c)^*(\mathcal{T}_m - \sigma I)^{-1}e_1\|b\|$$

Alternative: Rational Krylov (Grimme-Gallivan-VanDooren etc.)
choosing the poles unresolved issue (A nonsymmetric)

An example: CD Player, $|h(\sigma)| = |C^*_{:,i}(A - i\sigma I)^{-1}B_{:,j}|$



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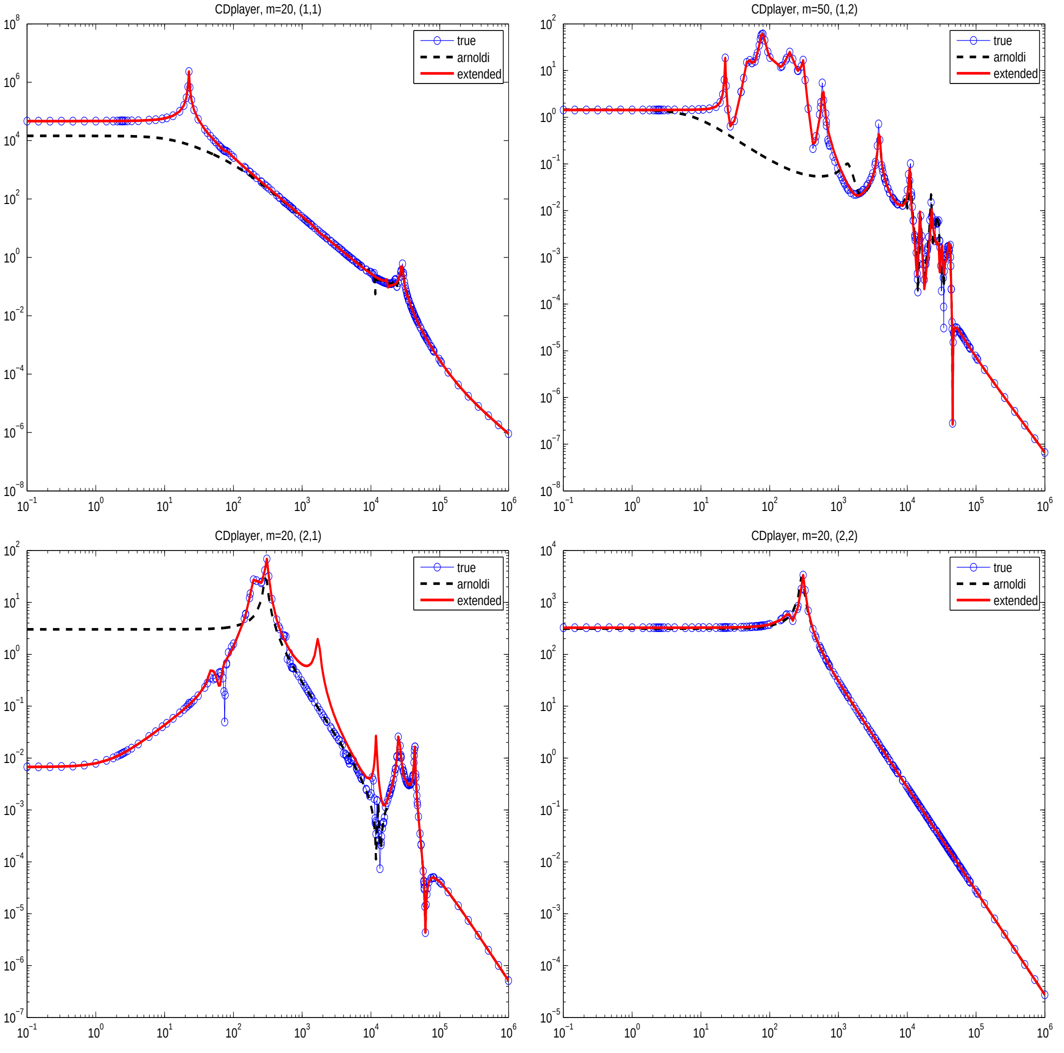


Other related problems

- Projected generalized Lyapunov equation
  $EXA^\top + AXE^\top = -P_l BB^\top P_l^\top, \quad X = P_r X P_r^\top$

(Stykel & Simoncini (in prep.))

- Sylvester equation: $AX + XB + C = 0$ (Heyouni '08)

- Riccati equation: $AX + XA^\top - XGX + C = 0$

(Heyouni & Jbilou '08)

- Shifted systems: $(A - \sigma I)x = b$ with many σ's

(..., Simoncini tr'09)

- Special Sylvester equation: $AX + X\Sigma = [b(\sigma_1), ..., b(\sigma_s)]$

(Simoncini tr'09)

Approximation space: Extended Krylov subspace

Balanced reduction.

Balancing matrix transformation. Given

$$AP + PA^\top + BB^\top = 0, \quad QA + A^\top Q + C^\top C = 0.$$

Find T_r, T_ℓ such that $T_\ell^\top P T_\ell = \Sigma = T_r^\top Q T_r$

The matrix

$$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3, \ldots)$$

contains the Hankel singular values of the system

Large body of literature, and various possibilities, even in the small-scale case (cf., e.g., Antoulas '05)

Error estimate for the reduced system:

$$\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|_{\mathcal{H}_\infty} \leq 2(\sigma_{k+1} + \cdots + \sigma_{\tilde{n}}),$$

An iterative procedure. Joint work in progress with T. Stykel

Given $\mathcal{K}_0$, $\mathcal{L}_0$.

For $k = 1, 2, \ldots$

1. Update approx. spaces $\mathcal{K}_{k-1} \to \mathcal{K}_k$=range($V_k$), $\mathcal{L}_{k-1} \to \mathcal{L}_k$=range($W_k$)
2. Compute approximate Gramians X_k, Y_k s.t.

$$P \approx P_k = V_k X_k V_k^\top, \quad Q \approx Q_k = W_k Y_k W_k^\top$$

   with $W_k^\top V_k = I$

3. Approximate Hankel singular values:

$$\sqrt{\lambda_j(PQ)} \approx \sigma_j(L_X^\top L_Y), \quad X_k = L_X L_X^\top, \quad Y_k = L_Y L_Y^\top$$

   $U\Sigma Z^\top =$ svd$(L_X^\top L_Y)$

4. If satisfied, compute truncated balancing transformation matrices:

$$T_r = V_k L_X U \Sigma^{-1/2}, \quad T_\ell = W_k L_Y Z \Sigma^{-1/2} \quad \text{and stop}$$

What spaces $\mathcal{K}_k$, $\mathcal{L}_k$ to obtain accurate and small size T_r, T_ℓ ?

Truncated balancing

What spaces $\mathcal{K}_k$, $\mathcal{L}_k$ to obtain accurate and small size T_r, T_ℓ ?

Two possible choices we are exploring:

$\star$ $\mathcal{K}_k = \mathcal{L}_k = \mathbf{EK}_k(A, [B, C^\top])$

(Related to cross-Gramians for A symmetric)

$\star$ $\mathcal{K}_k = \mathbf{EK}_k(A, B) \qquad \mathcal{L}_k = \mathbf{EK}_k(A^\top, C^\top)$

bi-orthogonal bases (à la Lanczos)

$\mathbf{EK}_k$: Extended Krylov subspace

Example

Penzl's example (408×408): $A =$ blkdiag(A_1, A_2, A_3, A_4, D)

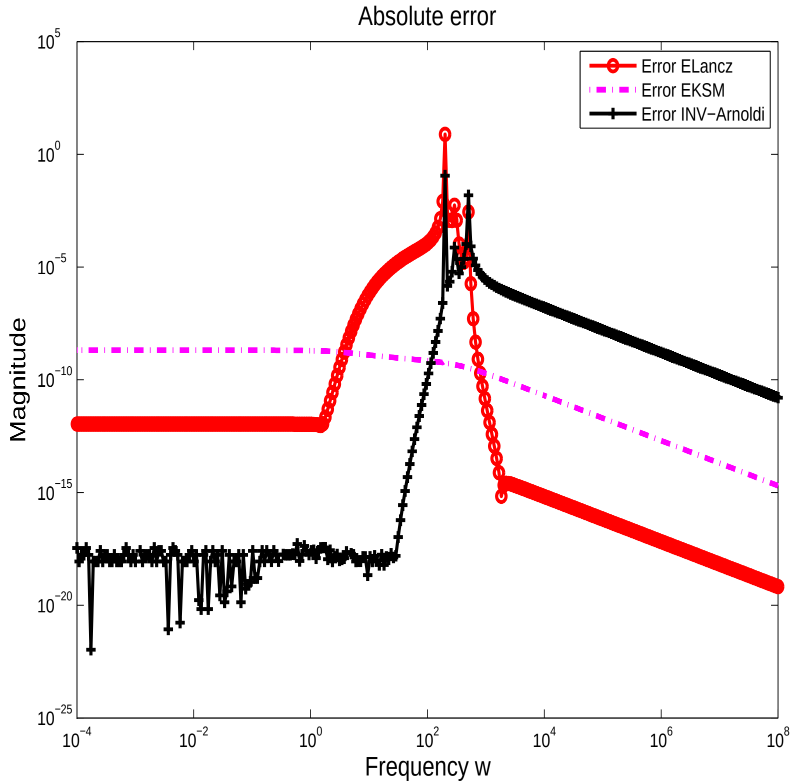
$$A_1 = \begin{bmatrix} -0.01 & -200 \\ 200 & 0.001 \end{bmatrix} A_2 = \begin{bmatrix} -0.2 & -300 \\ 300 & -0.1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.02 & -500 \\ 500 & 0 \end{bmatrix} A_4 = \begin{bmatrix} -0.01 & -520 \\ 520 & -0.01 \end{bmatrix}$$

and $D =$diag(1:400)

$B = C^\top$. Vector $(s = 1)$ with large projection onto nonsym part.

Example. cont'ed. Error $|H(\sigma) - H_k(\sigma)|$

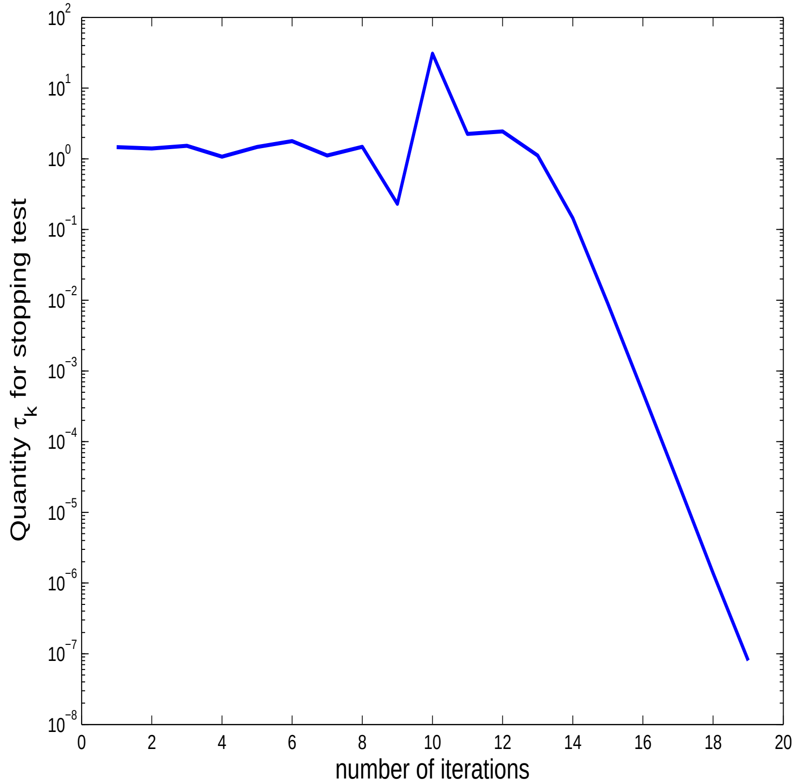


Balancing Ext'd Krylov: T_r of size 19 (space of max size 40)

Balancing Ext'd Lanczos: T_r of size 17 (left-right spaces of max size 20 each)

Inverted-Arnoldi: space of size 40

Convergence of Hankel singular values



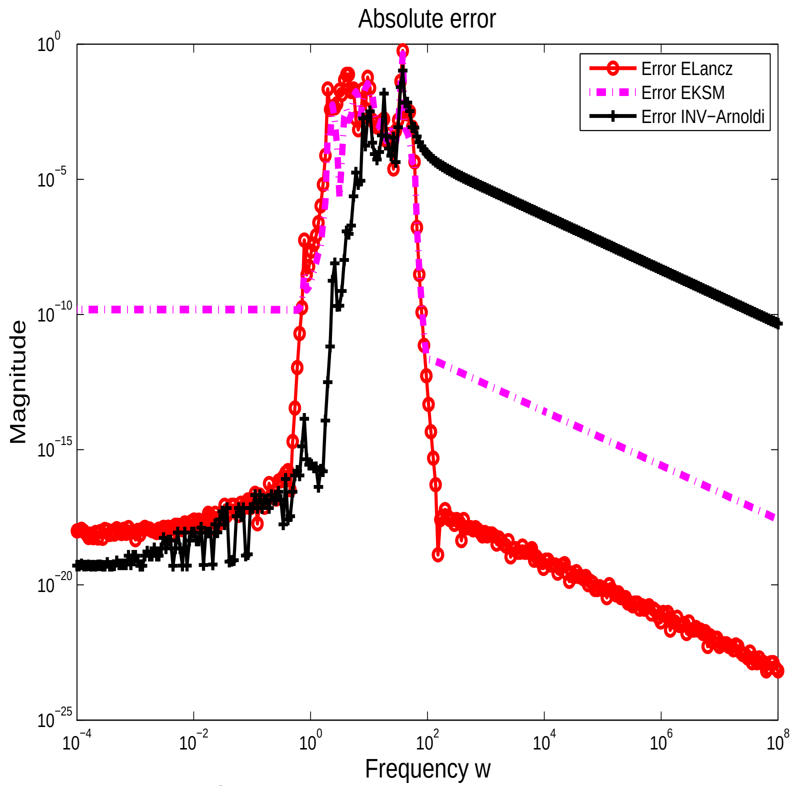
EK: At each iteration k

$$\tau_k = \sum_{j \in \mathcal{J}_k} \frac{|\sigma_j^{(k-1)} - \sigma_j^{(k)}|}{\sigma_1^{(k)}} \text{ where } \mathcal{J}_k = \{\text{index } j \, : \, \sigma_j/\sigma_1 > 10^{-10}\}$$

Residuals of Lyapunov equations: $\|R_k\| = \|S_k\| = O(10^{-3})$

One more example. Error $|H(\sigma) - H_k(\sigma)|$

ISS case. (tiny: 270×270), $B \neq C^\top$ vectors

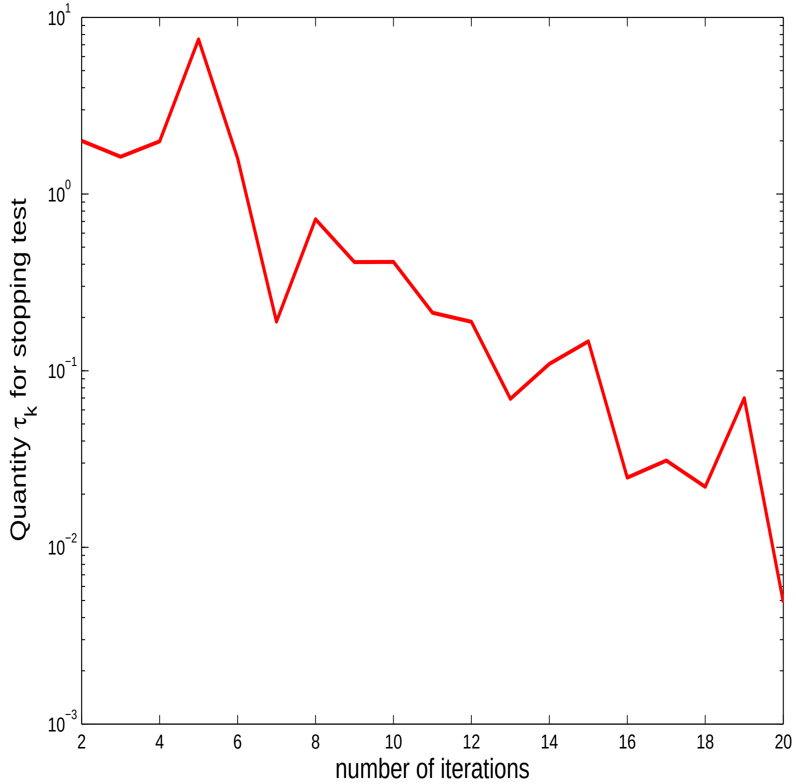


Balancing Ext'd Krylov: T_r of size 35 (space of max size 40)

Balancing Ext'd Lanczos: T_r of size 20 (left-right spaces of max size 20 each)

Inverted-Arnoldi: space of size 40

Convergence of Hankel singular values



EK Lanczos: At each iteration k

$$\tau_k = \sum_{j \in \mathcal{J}_k} \frac{|\sigma_j^{(k-1)} - \sigma_j^{(k)}|}{\sigma_1^{(k)}} \text{ where } \mathcal{J}_k = \{\text{index } j \, : \, \sigma_j/\sigma_1 > 10^{-10}\}$$

Ext'd Lanczos. Residuals of Lyapunov equations: $\|R_k\| = O(1), \;\; \|S_k\| = O(10^3)$

Conclusions

- Great potential of enriched projection spaces
- Exploit low cost of using A and A^{-1}
- Projection combined with matrix function theory for proofs

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References

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2. V. Simoncini and Tatjana Stykel, *Iterative methods for balanced truncation of large-scale linear dynamical systems*, In preparation.

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5. V. Simoncini, *A new iterative method for solving large-scale Lyapunov matrix equations*, SIAM J. Scient. Computing, v.29, n.3 (2007), pp. 1268-1288.