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Convergence properties of preconditioned iterative solvers for saddle point linear systems

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The Problem

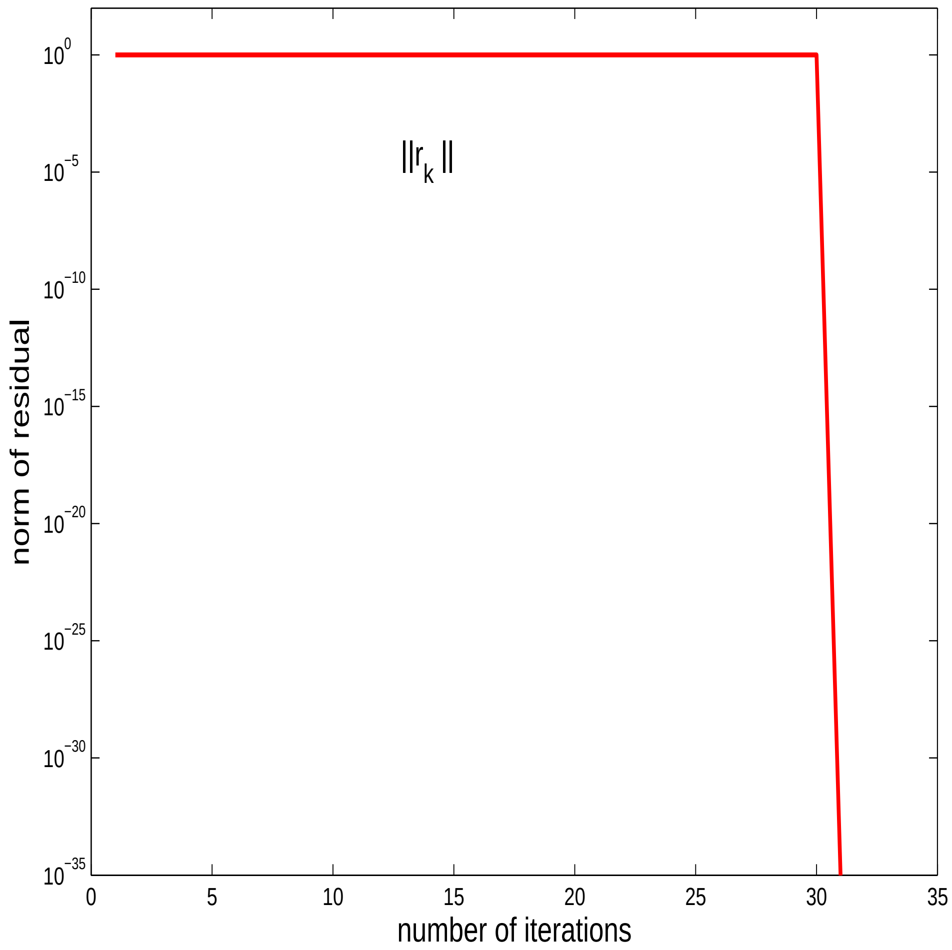
Given

$$Ax = b$$

$A \in \mathbb{R}^{n \times n}$ nonsymmetric (in general, already preconditioned)

Derive sufficient conditions for non-stagnation of GMRES-type solvers

That is, whether we can predict that



does not occur! $(31 \times 31$ matrix)

Motivation

Complete stagnation is a very unfortunate but rare event

Other reasons for studying this problem:

Motivation

Complete stagnation is a very unfortunate but rare event

Other reasons for studying this problem:

Let x_k be an approximate solution, and $r_k = b - Ax_k$.

- Partial stagnation phases occur more frequently (staircase slope)
- Bounds of the type

$$\|r_{k+1}\| \le c\|r_k\|, \quad 0 < c < 1$$

important whenever c independent of problem parameters

$\Rightarrow$ convergence behavior is not influenced by other model components: $\|r_k\| \le c^k\|r_0\|$

$\Rightarrow$ Crucial to design preconditioning techniques

Elman bound (PhD thesis, 1982)

Let $H = (A + A^T)/2$

If H is positive definite (i.e. $\lambda_{\min}(H) > 0$), then

$$\|r_k\| \leq \left(1 - \frac{\lambda^2_{\min}(H)}{\|A\|^2}\right)^{\frac{1}{2}} \|r_{k-1}\| < \|r_{k-1}\|$$

so that

$$\|r_k\| \leq \left(1 - \frac{\lambda^2_{\min}(H)}{\|A\|^2}\right)^{\frac{k}{2}} \|r_0\|$$

Non-Stagnation and Parameter independence

$$\|r_k\| \leq \left(1 - \frac{\lambda_{\min}^2(H)}{\|A\|^2}\right)^{\frac{k}{2}} \|r_0\|$$

If $\lambda_{\min}(H), \|A\|$ independent of parameters (viscosity, meshsize, etc.):

Number of iterations to converge is independent of parameters

- Bound *per se* is not sharp
- Very much used in certain contexts

  (e.g. Domain Decomposition methods, cf. Toselli & Widlund 2005)

Related and unrelated bounds

After one iteration of a minimal residual method:

$$\|r_1\| = \sqrt{1 - \frac{(r_0^T A r_0)^2}{\|Ar_0\|^2 \, \|r_0\|^2}}\,\|r_0\|$$

...true stagnation is $very$ unlikely !

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- Characterization of matrices which lead to complete stagnation

  (Zavorin etal. 2003)

- Some improvements over this bound for diag.ble/nondiag.ble matrices

  (Eisenstat etal. '83, Greenbaum '97, Saad '03, Liesen '00, Freund '90, ...)

- Different bounds, using $\mathcal{F}(A) \subset \mathbb{C}^+$

  (Eiermann & Ernst '01, Greenbaum '97, Starke '97)

- Additional results for A normal (s.t. $AA^T = A^T A$)

The new non-stagnation condition

Grcar tr'89:

Let q_k be polynomial with $q_k(0) = 0$. If $\frac{1}{2}(q_k(A) + q_k(A)^T) > 0$ then

$$\|r_k\| \leq \left(1 - \frac{\theta_{\min}^2}{\|q_k(A)\|^2}\right)^{\frac{1}{2}} \|r_0\| \quad \theta_{\min} = \lambda_{\min}(\frac{1}{2}(q_k(A) + q_k(A)^T))$$

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Finding such a q_k is not simple!

We reverse the problem:

We fix $q_k(t) = t^k$, $k = 2, 4$ and determine conditions on A such that Grcar's result can be applied

Sufficient condition

For $q_k(t) = t^k$, $k = 2$:

If A is such that Grcar's result holds, then GMRES cannot stagnate for more than $k - 1 = 1$ consecutive iterations

(Similar for $k = 4$)

Note: Also relevant for restarted GMRES

DEF. M is positive definite if $\frac{1}{2}(M + M^T) > 0$

Restatement of the problem:

Find conditions on A *so that* $q_2(A) = A^2$ *is positive definite*

The new conditions

Let $H = \frac{1}{2}(A + A^T)$, $S = \frac{1}{2}(A - A^T)$.

1. If H is nonsingular, then A^2 is positive definite if and only if

$$\|SH^{-1}\| < 1$$

2. If S is nonsingular, then A^2 is negative definite if and only if

$$\|HS^{-1}\| < 1$$

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$$\|r_2\| \leq \left(1 - \frac{\theta_{\min}^2}{\|A^2\|^2}\right)^{\frac{1}{2}} \|r_0\| \quad \theta_{\min} = \lambda_{\min}(\frac{1}{2}(A^2 + (A^2)^T)) > 0$$

The same relation holds at every other iteration

A simple Sufficient condition

H "dominates" S:

If $\min_i |\lambda_i(H)| > \max_j |\lambda_j(S)|$, then A^2 is positive definite

(A corresponding result for A^2 negative definite)

The $k = 4$ case

Let $H = \frac{1}{2}(A + A^T)$, $S = \frac{1}{2}(A - A^T)$.

1. If $H^2 + S^2$ is nonsingular, then A^4 is positive definite if and only if

$$\|(HS + SH)(H^2 + S^2)^{-1}\| < 1$$

2. If $HS + SH$ is nonsingular, then A^4 is negative definite if and only if

$$\|(H^2 + S^2)(HS + SH)^{-1}\| < 1$$

⋆ One could continue with higher powers, but ....

⋆ There may be other polynomials $q_k(t)$ such that Grcar's result applies

Some Examples

FD discretization of:

$$L(u) = -(\alpha u_{x_1})_{x_1} - (\beta u_{x_2})_{x_2} + \gamma u_{x_1} + \delta u_{x_2} - \eta u$$

size(A) =1600. $\eta = 100$.

α	β	γ	δ	$\lambda_{\min}(H)$	$\|SH^{-1}\|$
$\exp(-x_1 x_2)$	$\exp(x_1 x_2)$	-1	-1	-0.04719	0.6194
1	1	$-1/(.1x_1 + 100x_2)$	0	-0.04775	0.1577
1	1	$1/10(x_1 - x_2)$	0	-0.04772	0.1838
1	1	$1/10(x_1 + x_2)$	0	-0.04772	0.5819
1	1	0.2	0	-0.04781	0.5811

Navier-Stokes problem. Flow over a backward facing step

IFISS Package (Elman, Ramage, Silvester)

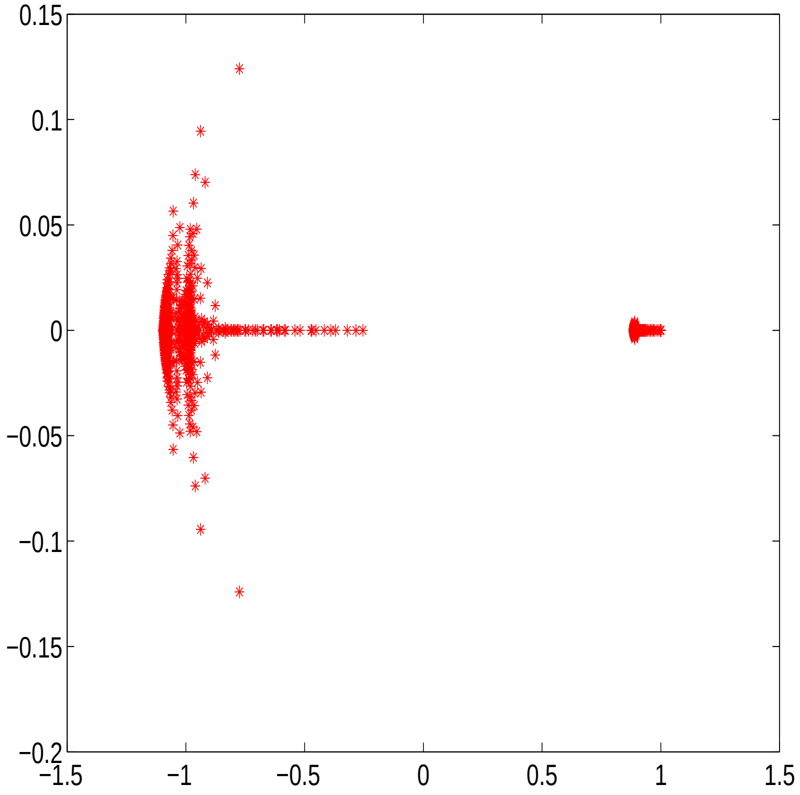
Oseen Problem. Uniform grid, Q1-P0 elements, F nonsymmetric

Augmentation block diagonal preconditioning:

$$\mathcal{A} = \begin{bmatrix} F & B^T \\ B & -\beta C \end{bmatrix} \quad \mathcal{P} = \begin{bmatrix} F + B^T \tilde{C}^{-1} B & \\ & \tilde{C} \end{bmatrix}$$

Spectrum of $\mathcal{A}\mathcal{P}^{-1}$ tends to cluster around $\lambda = 1$, $\lambda = -1$ (Cao, 2008)

Spectrum and condition



$n = 418, m = 176, \; \|SH^{-1}\| = 0.99856 < 1$

$n = 1538, m = 704, \; \|SH^{-1}\| = 0.99568 < 1$

Stokes Problem. Channel domain

IFISS Package (Elman, Ramage, Silvester)

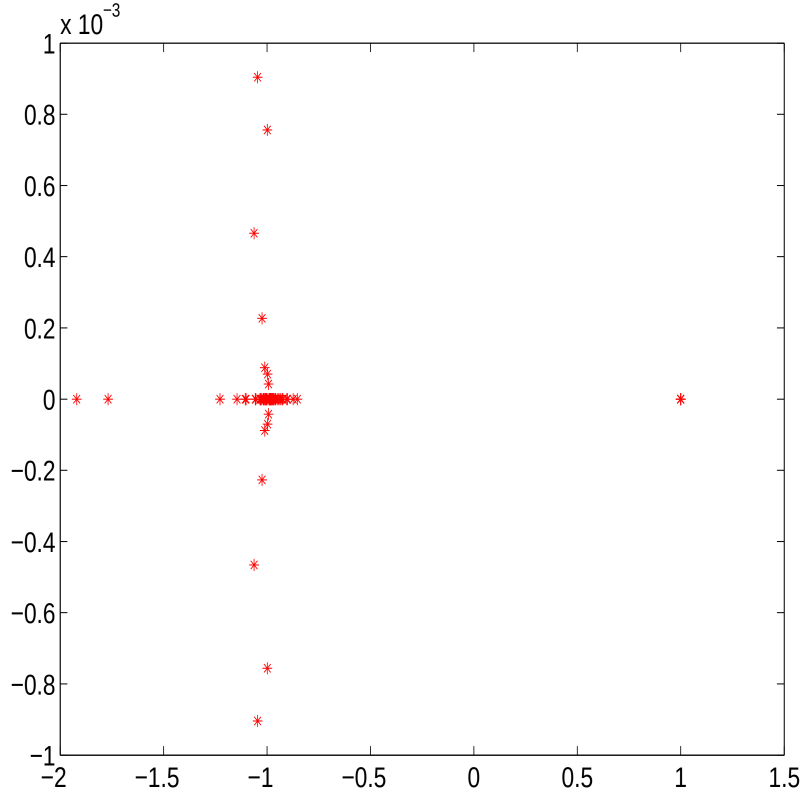
uniform grid, Q1-P0 elements, M symmetric

Nonsymmetric Preconditioning (cf. Elman, Silvester & Wathen '05):

$$\mathcal{A} = \begin{bmatrix} M & B^T \\ B & \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} M & B^T \\ & G \end{bmatrix}, \quad G \approx BM^{-1}B$$

Spectrum of $\mathcal{A}\mathcal{P}^{-1}$ tends to cluster around $\lambda = 1, -1$

Spectrum and condition



$\Lambda(H) = [-1.97,\ 1.03]$

$n = 162, m = 64,\ G = B\widetilde{M}^{-1}B \qquad \|SH^{-1}\| = 0.3218 < 1$

$n = 578, m = 256,\ G = B\widetilde{M}^{-1}B \qquad \|SH^{-1}\| = 0.6399 < 1$

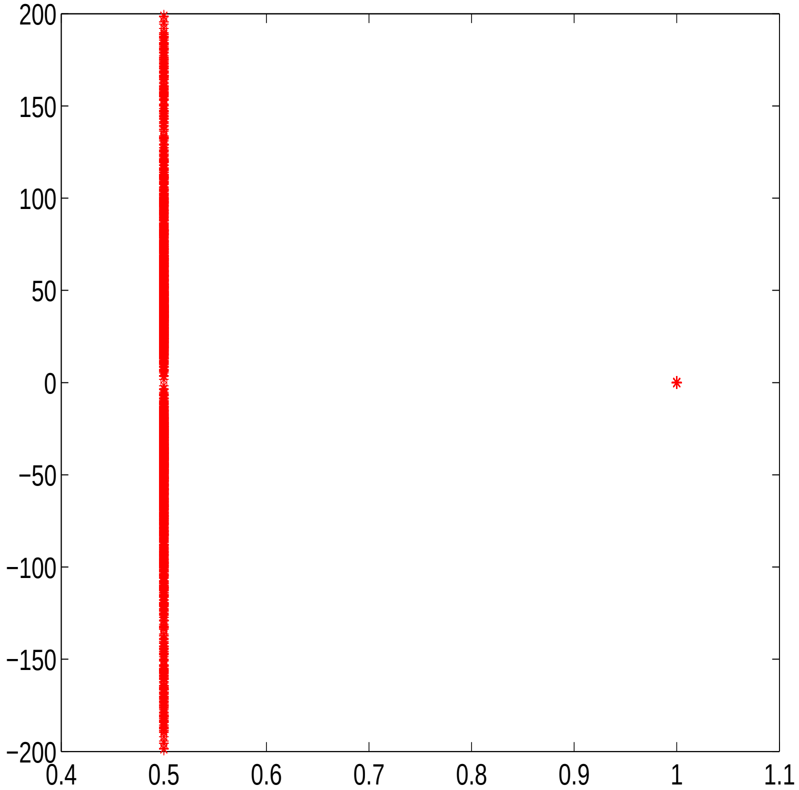
Symmetric Saddle-Point type Problem

Nonsymmetric version (cf. survey: Benzi, Golub & Liesen '05):

$$\mathcal{A}_- = \begin{bmatrix} \mu I & B^T \\ -B & 0 \end{bmatrix}, \quad \mu > 0$$

Spectrum of $\mathcal{A}_-$ is in $\mathbb{C}^+$, but $\frac{1}{2}(\mathcal{A}_- + \mathcal{A}_-^T) \geq 0$

Spectrum and condition



$\mu = 1 \qquad \Lambda(H) = [0,\ 1]$

$n = 1272, m = 816,\ \|(HS + SH)(H^2 + S^2)^{-1}\| = 0.7856 < 1$

Conclusions

- New conditions for non-stagnation:

  Useful to establish parameter independence

- Possibility to extend the result

REFERENCE

V. Simoncini and Daniel B. Szyld

New conditions for non-stagnation of minimal residual methods

Numerische Mathematik, v. 109, n.3 (2008), pp. 477-487