Precondizionatori Strutturati per Sistemi algebrici di Punto Sella

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Application problems

- Computational Fluid Dynamics
- Elasticity problems
- Mixed (FE) formulations of II and IV order elliptic PDEs
- Linearly Constrained Programs
- Linear Regression in Statistics
- Weighted Least Squares (Image restoration)
- ... Survey: Benzi, Golub and Liesen, Acta Num 2005

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

Constrained Quadratic minimization problem

$$\text{minimize} \quad J(u) = \frac{1}{2}\langle Au, u\rangle - \langle f, u\rangle$$

subject to $Bu = g$

$A \quad n \times n$ spd, $B \; m \times n$, $m \leq n$ full rank

$$\Downarrow$$

Lagrange multipliers approach

Karush-Kuhn-Tucker (KKT) system

Spectral properties

- $\mathcal{M} = \begin{bmatrix} A & B^T \\ B & O \end{bmatrix}$ $\quad$ $0 < \lambda_n \le \cdots \le \lambda_1$ eigs of A
  $0 < \sigma_m \le \cdots \le \sigma_1$ sing. vals of B

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- (Rusten & Winther 1992) $\Lambda(\mathcal{M})$ subset of

$$\left[\frac{1}{2}(\lambda_n - \sqrt{\lambda_n^2 + 4\sigma_1^2}), \frac{1}{2}(\lambda_1 - \sqrt{\lambda_1^2 + 4\sigma_m^2})\right] \quad \cup \quad \left[\lambda_n, \frac{1}{2}(\lambda_1 + \sqrt{\lambda_1^2 + 4\sigma_m^2})\right]$$

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- (Silvester & Wathen 1994), $0 \leq \sigma_m \leq \cdots \leq \sigma_1$

$$\lambda_n - \lambda_{\max}(C) - \sqrt{(\lambda_n + \lambda_{\max}(C))^2 + 4\sigma_1^2}$$

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$$\lambda_n - \lambda_{\max}(C) - \sqrt{(\lambda_n + \lambda_{\max}(C))^2 + 4\sigma_1^2}$$

More results for special cases (e.g., Perugia & S. 2000)

Block diagonal Preconditioner

$$\mathcal{P} = \begin{bmatrix} \widetilde{A} & 0 \\ 0 & \widetilde{C} \end{bmatrix} \qquad \text{sym. pos. def.}$$

$$\widetilde{A} \approx A \qquad \widetilde{C} \approx BA^{-1}B^T + C$$

Rusten Winther (1992), Silvester Wathen (1993-1994), Klawonn (1998)
Fischer Ramage Silvester Wathen (1998...), . . .

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$\lambda \neq 0$ eigs of $\mathcal{P}^{-\frac{1}{2}}\mathcal{M}\mathcal{P}^{-\frac{1}{2}}$,

$$\lambda \in [-a, -b] \cup [c, d], \qquad a, b, c, d > 0$$

Triangular preconditioner

$$A \text{ spd}, \quad \mathcal{P} = \begin{bmatrix} \widetilde{A} & B^T \\ 0 & -\widetilde{C} \end{bmatrix}$$

Bramble & Pasciak, Elman, Klawonn, Axelsson & Neytcheva, S. 2004

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Spectrum of $\mathcal{M}\mathcal{P}^{-1}$:

small complex cluster around 1 + real interval

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Spectrum of $\mathcal{M}\mathcal{P}^{-1}$:

small complex cluster around 1 + real interval

More precisely: $\theta \in \Lambda(\mathcal{M}\mathcal{P}^{-1})$

$\Im(\theta) \neq 0 \Rightarrow |\theta - 1| \leq \sqrt{1 - \lambda_{\min}(A\widetilde{A}^{-1})}$ (if $1 - \lambda_{\min}(A\widetilde{A}^{-1}) \geq 0$)

$\Im(\theta) = 0 \Rightarrow \theta \in [\chi_1, \chi_2] \ni 1$

Constraint Preconditioner

$$\mathcal{Q} = \left[\begin{array}{cc} \widetilde{A} & B^T \\ B & -C \end{array} \right]$$

Axelsson (1979), Ewing Lazarov Lu Vassilevski (1990), ... many more papers 1997 -

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$\lambda \neq 0$ eigs of $\mathcal{M}\mathcal{Q}^{-1}$, $\lambda \in \mathbb{R}^+$, $\lambda \in \{1\} \cup [\alpha_0, \alpha_1]$

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$$\mathcal{Q}^{-1} = \begin{bmatrix} I & -B^T \\ O & I \end{bmatrix} \begin{bmatrix} I & O \\ O & -(\mathbf{BB^T} + \mathbf{C})^{-1} \end{bmatrix} \begin{bmatrix} I & O \\ -B & I \end{bmatrix}$$

($\widetilde{A} = I$ if prescaling used)

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($\widetilde{A} = I$ if prescaling used)

Results on eigenvectors (Dollar tr05)

More general factorizations (Dollar& Gould & Schilders & Wathen 06, Bergamaschi & Gondzio & Venturin & Zilli, tr05)

Computational Considerations.

3D Magnetostatic problem

$H \approx BB^T + C$ with H: Incomplete Cholesky fact.

(ICT package, Saad & Chow)

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Elapsed Time

size	MA27	QMR $\mathcal{Q}$	QMR $\widehat{\mathcal{Q}}(2)(it)$	QMR ILDLT(10)	
1119	**0.6**	**3.0**	**1.7**(18)	**0.7**	
2208	**2.3**	**11.7**	**3.1**(18)	**1.5**	
4371	**10.2**	**64.6**	**8.4**(20)	**5.2**	
8622	**83.4**	**466.0**	**18.3**(29)	**31.0**	
22675	**753.5**	**3745.5**	**63.2**(45)	**246.0**	

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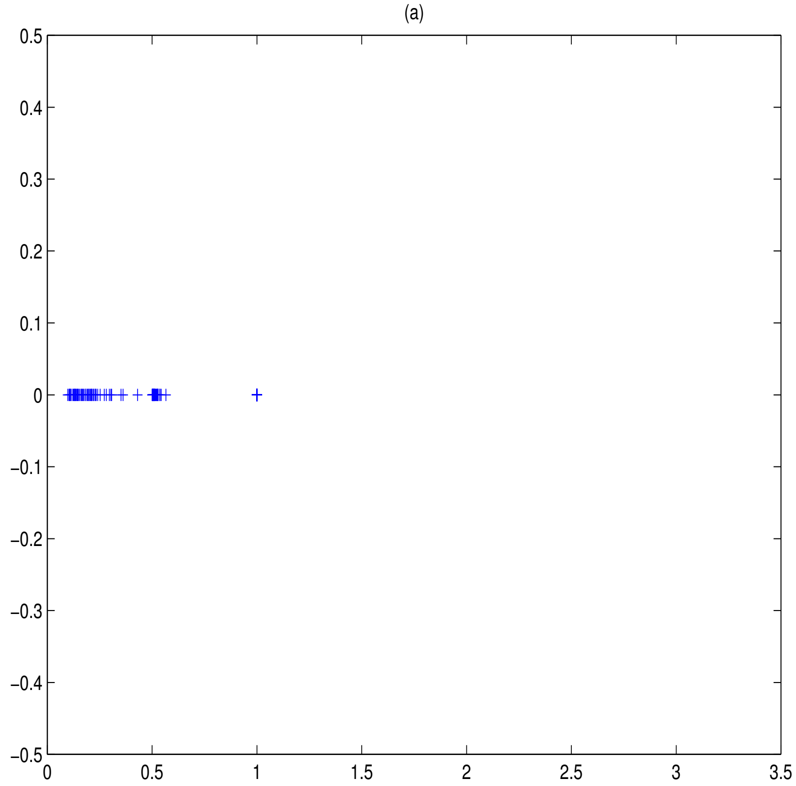
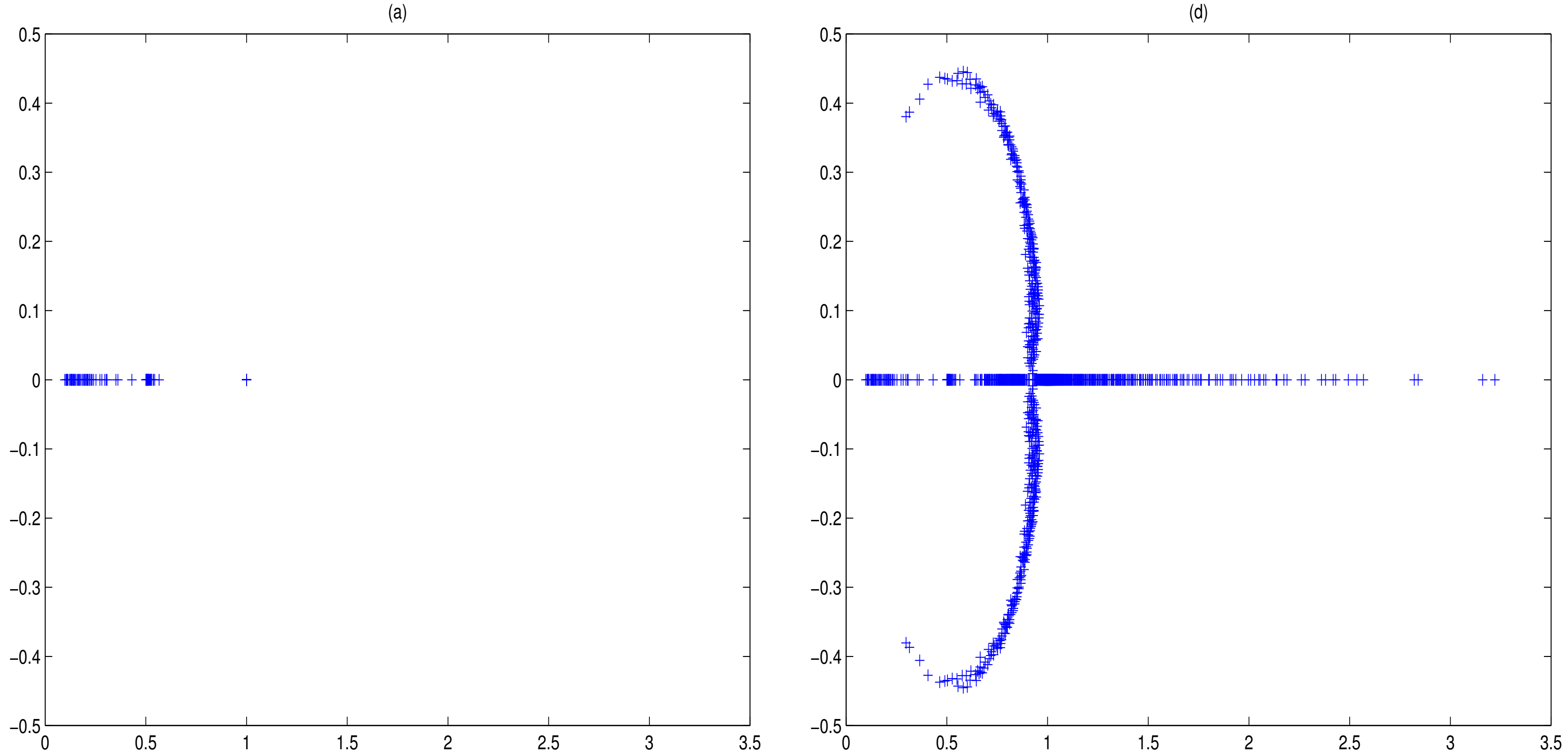
size	MA27	QMR $\mathcal{Q}$	QMR $\widehat{\mathcal{Q}}(2)(it)$	QMR ILDLT(10)	QMR $\mathcal{P}$	QMR $\widehat{\mathcal{P}}(2)(it)$
1119	**0.6**	**3.0**	**1.7**(18)	**0.7**	**4.0**	**2.0**
2208	**2.3**	**11.7**	**3.1**(18)	**1.5**	**15.2**	**4.9**
4371	**10.2**	**64.6**	**8.4**(20)	**5.2**	**73.9**	**11.0**
8622	**83.4**	**466.0**	**18.3**(29)	**31.0**	**510.1**	**24.3**
22675	**753.5**	**3745.5**	**63.2**(45)	**246.0**	**4161.4**	**128.2**

3D Magnetostatic problem



Spectrum of perturbed problem

3D Magnetostatic problem



$$\|(BB^T + C) - H\|_\infty \approx 2 \cdot 10^{-1} \|BB^T + C\|_\infty$$

Eigenvalue problem

$$\widehat{\mathcal{Q}} = \begin{bmatrix} I & O \\ B & I \end{bmatrix} \begin{bmatrix} I & O \\ O & -H \end{bmatrix} \begin{bmatrix} I & B^T \\ O & I \end{bmatrix}, \quad H \approx BB^T, \quad C = O$$

$$\begin{bmatrix} A & B^T \\ B & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \widehat{\mathcal{Q}} \begin{bmatrix} x \\ y \end{bmatrix}$$

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$$\begin{bmatrix} A & B^T \\ B & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \widehat{\mathcal{Q}} \begin{bmatrix} x \\ y \end{bmatrix}$$

can be written as

$$\begin{bmatrix} I & O \\ -B & I \end{bmatrix} \begin{bmatrix} A & B^T \\ B & O \end{bmatrix} \begin{bmatrix} I & -B^T \\ O & I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} I & O \\ O & -H \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} A & (I-A)B^T \\ B(I-A) & -B(2I-A)B^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} I & O \\ O & -H \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

A “different” perspective

$$\begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix} \qquad \mathcal{M}_- x = d$$

Polyak 1970, ..., Wathen & Fischer & Silvester 1995, Fischer & Ramage & Silvester & Wathen 1998, Fischer & Peherstorfer 2001, Bai & Golub & Ng 2003, Sidi 2003, Benzi & Gander & Golub 2003, Benzi & Golub 2004, S. & Benzi 2004, ...

A “different” perspective

$$\begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix} \qquad \mathcal{M}_{-} x = d$$

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- $\mathcal{M}$ positive real $\quad \Rightarrow \quad \Lambda(\mathcal{M}_{-})$ in $\quad \mathbb{C}^{+}$

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- $\mathcal{M}$ positive real $\quad \Rightarrow \quad \Lambda(\mathcal{M}_-)$ in $\mathbb{C}^+$

- More refined spectral information possible

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- $\mathcal{M}$ positive real $\quad \Rightarrow \quad \Lambda(\mathcal{M}_-)$ in $\mathbb{C}^+$

- More refined spectral information possible
- New classes of preconditioners
- General framework for spectral analysis of some indefinite preconditioners

Benzi & S. (Num.Math 2006)

Spectral properties of $\mathcal{M}_-$

$$\mathcal{M}_- = \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix}$$

A $\quad n \times n$ sym. semidef. matrix, $\qquad B$ $\quad m \times n$, $m \leq n$

$\mathcal{M}_-$ has at least $n - m$ real eigenvalues

Detailed analysis for $A = \eta I, C = O$ in:

- $\star$ Fischer & Ramage & Silvester & Wathen 1998
- $\star$ Fischer & Peherstorfer 2001

Reality condition

Let A be spd and $C = \beta I$, β real. If

$$\lambda_{\min}(A + \beta I) \geq 4\,\lambda_{\max}(B^T A^{-1} B + \beta I),$$

then all eigenvalues of $\mathcal{M}_-$ are real (and positive)

Benzi & S. 2006

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Benzi & S. 2006

The condition is sufficient, but not necessary:

$$\mathcal{M}_- = \left[\begin{array}{cc|c} \frac{1}{2} & 0 & 0 \\ 0 & 3 & 1 \\ \hline 0 & -1 & 0 \end{array}\right], \qquad \begin{array}{l} \lambda_{\min}(A) = \frac{1}{2} \\ \lambda_{\max}(BA^{-1}B) = \frac{1}{3} \end{array}$$

has real spectrum, but does not satisfy the condition

The (steady-state) Stokes problem

$$\left\{\begin{array}{ll} -\Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \\ \mathcal{B}\mathbf{u} = \mathbf{g} & \text{on } \Gamma\,. \end{array}\right. \qquad \Rightarrow \qquad \mathcal{M}_{\pm}$$

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Numerical evidence: eigs of $\mathcal{M}_-$ are all real and positive

(for several discretizations and b.c.)

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(for several discretizations and b.c.)

Explanation:

$\Omega = [0,1] \times [0,1]$, Dirichlet b.c. div-stable discr.

$$\begin{aligned} &\lambda_{\min}(A) \approx 2\pi^2 \approx 19, \\ &\lambda_{\max}(BA^{-1}B^T) \approx 1 \end{aligned} \qquad \Rightarrow \lambda_{\min}(A) > 4\lambda_{\max}(BA^{-1}B^T)$$

(In)definite inner product

$$\mathcal{M}_- = \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \qquad J = \begin{bmatrix} I & O \\ O & -I \end{bmatrix}$$

J indefinite

$\mathcal{M}_-$ is J-sym.

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If $\Lambda(\mathcal{M}_-) \subset \mathbb{R}^+$, there exists a nonstandard inner product with which $\mathcal{M}_-$ is spd (and diag.ble)

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If $\Lambda(\mathcal{M}_-) \subset \mathbb{R}^+$, there exists a nonstandard inner product with which $\mathcal{M}_-$ is spd (and diag.ble)

$$G = \begin{bmatrix} A - \gamma I & B^T \\ B & \gamma I \end{bmatrix}, \qquad \mathcal{M}_- G = G \mathcal{M}_-^T$$

(In)definite inner product

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If $\Lambda(\mathcal{M}_-) \subset \mathbb{R}^+$, there exists a nonstandard inner product with which $\mathcal{M}_-$ is spd (and diag.ble)

$$G = \begin{bmatrix} A - \gamma I & B^T \\ B & \gamma I \end{bmatrix}, \qquad \mathcal{M}_- G = G \mathcal{M}_-^T$$

If $\gamma = \frac{1}{2}\lambda_{\min}(A)$ and $\lambda_{\min}(A) > 4\lambda_{\max}(BA^{-1}B^T)$ (with $C = O$)

$\Rightarrow \quad G$ is spd $\quad \Rightarrow \quad G$ defines an inner product for $\mathcal{M}_-$

Benzi & S. 2006. Case of general C recently covered (Liesen tr 2006)

Hermitian Skew-Hermitian Preconditioning

$$\mathcal{M}_- = \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} = \begin{bmatrix} A & O \\ O & C \end{bmatrix} + \begin{bmatrix} O & B^T \\ -B & O \end{bmatrix}$$

$$= \quad \mathcal{H} \quad + \quad \mathcal{S}$$

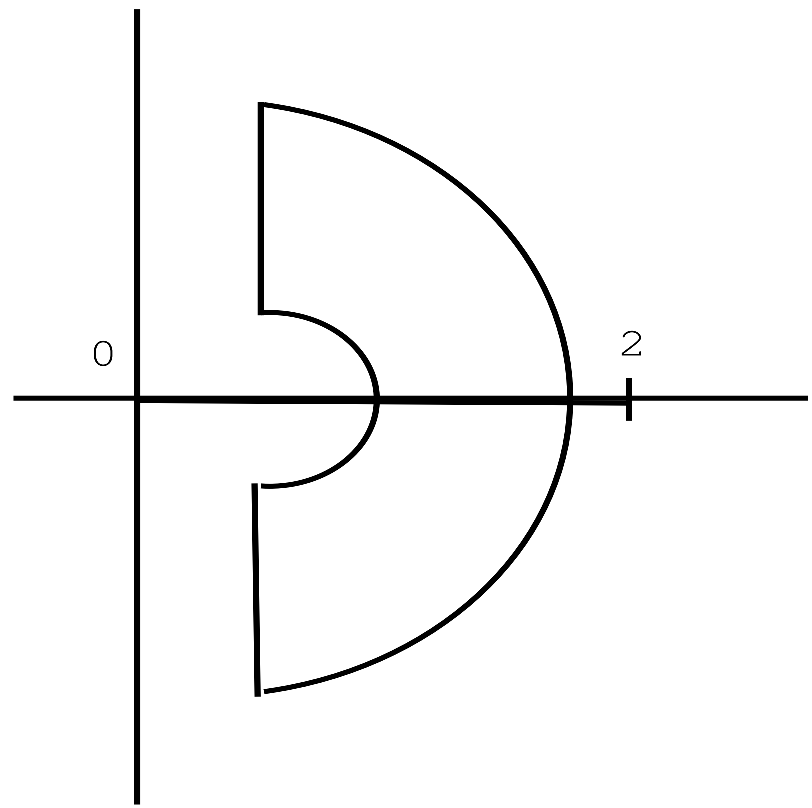
$$\mathcal{M}_- = \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} = \begin{bmatrix} A & O \\ O & C \end{bmatrix} + \begin{bmatrix} O & B^T \\ -B & O \end{bmatrix}$$
$$= \quad \mathcal{H} \quad + \quad \mathcal{S}$$

Use the preconditioner

$$\mathcal{R}_\alpha = \frac{1}{2\alpha}(\mathcal{H} + \alpha I)(\mathcal{S} + \alpha I) \qquad \alpha \in \mathbb{R},\ \alpha > 0$$

(Bai & Golub & Ng 2003), Benzi & Gander & Golub 2003, Benzi & Golub 2004, S. & Benzi 2004, Benzi & Ng, 2004

Spectrum of $\mathcal{R}_\alpha^{-1}\mathcal{M}_-$ (case $C = 0$)



(S. & Benzi 2004)

Reality condition

Assume A is spd and $C = 0$. If

$$\alpha \le \frac{1}{2}\lambda_{\min}(A)$$

then all eigenvalues of $\mathcal{R}_\alpha^{-1}\mathcal{M}_-$ are real (S. & Benzi '04)

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then all eigenvalues of $\mathcal{R}_\alpha^{-1}\mathcal{M}_-$ are real (S. & Benzi '04)

But even more general:

If $\lambda_{\min}(A) > 4\lambda_{\max}(BA^{-1}B^T)$ then

all eigenvalues are real for any α

(cf. Stokes problem)

Location of $\mathcal{M}_-$'s spectrum

Let $\lambda \in \Lambda(\mathcal{M}_-)$, $x = [u; v]$ e'vect., A spd (*cf. Sidi '03,* $C = O$)

$\star$ If $\Im(\lambda) \neq 0$ then

$$
\begin{aligned}
\frac{1}{2}(\lambda_{\min}(A) + \lambda_{\min}(C)) \leq \quad \Re(\lambda) \quad & \leq \frac{1}{2}(\lambda_{\max}(A) + \lambda_{\max}(C)) \\
|\Im(\lambda)| \quad & \leq \sigma_{\max}(B).
\end{aligned}
$$

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$\star$ If $\Im(\lambda) = 0$ then for $v \neq 0$

$$\min\{\lambda_{\min}(A), \lambda_{\min}(C)\} \leq \lambda \leq \max\{\lambda_{\max}(A), \lambda_{\max}(C)\}$$

and for $v = 0$, $\lambda_{\min}(A) \leq \lambda \leq \lambda_{\max}(A)$

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$$\begin{aligned} \frac{1}{2}(\lambda_{\min}(A) + \lambda_{\min}(C)) \leq \quad & \Re(\lambda) & \leq \frac{1}{2}(\lambda_{\max}(A) + \lambda_{\max}(C)) \\ & |\Im(\lambda)| & \leq \sigma_{\max}(B). \end{aligned}$$

$\star$ If $\Im(\lambda) = 0$ then for $v \neq 0$

$$\min\{\lambda_{\min}(A), \lambda_{\min}(C)\} \leq \lambda \leq \max\{\lambda_{\max}(A), \lambda_{\max}(C)\}$$

and for $v = 0$, $\lambda_{\min}(A) \leq \lambda \leq \lambda_{\max}(A)$

$$\mathcal{M}_- = \left[\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 1 & 0 \\ \hline -1 & 0 & 1 \end{array}\right] \qquad \lambda_1 = 1, \quad \lambda_{2,3} = \frac{3}{2} \pm i\frac{\sqrt{3}}{2}.$$

$$\begin{bmatrix} A & (I-A)B^T \\ B(I-A) & -B(2I-A)B^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} I & O \\ O & -H \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} A & (I-A)B^T \\ B(I-A) & -B(2I-A)B^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} I & O \\ O & -H \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Eigenvalue bounds: Let $\widehat{C} = B(2I-A)B^T H^{-1}$

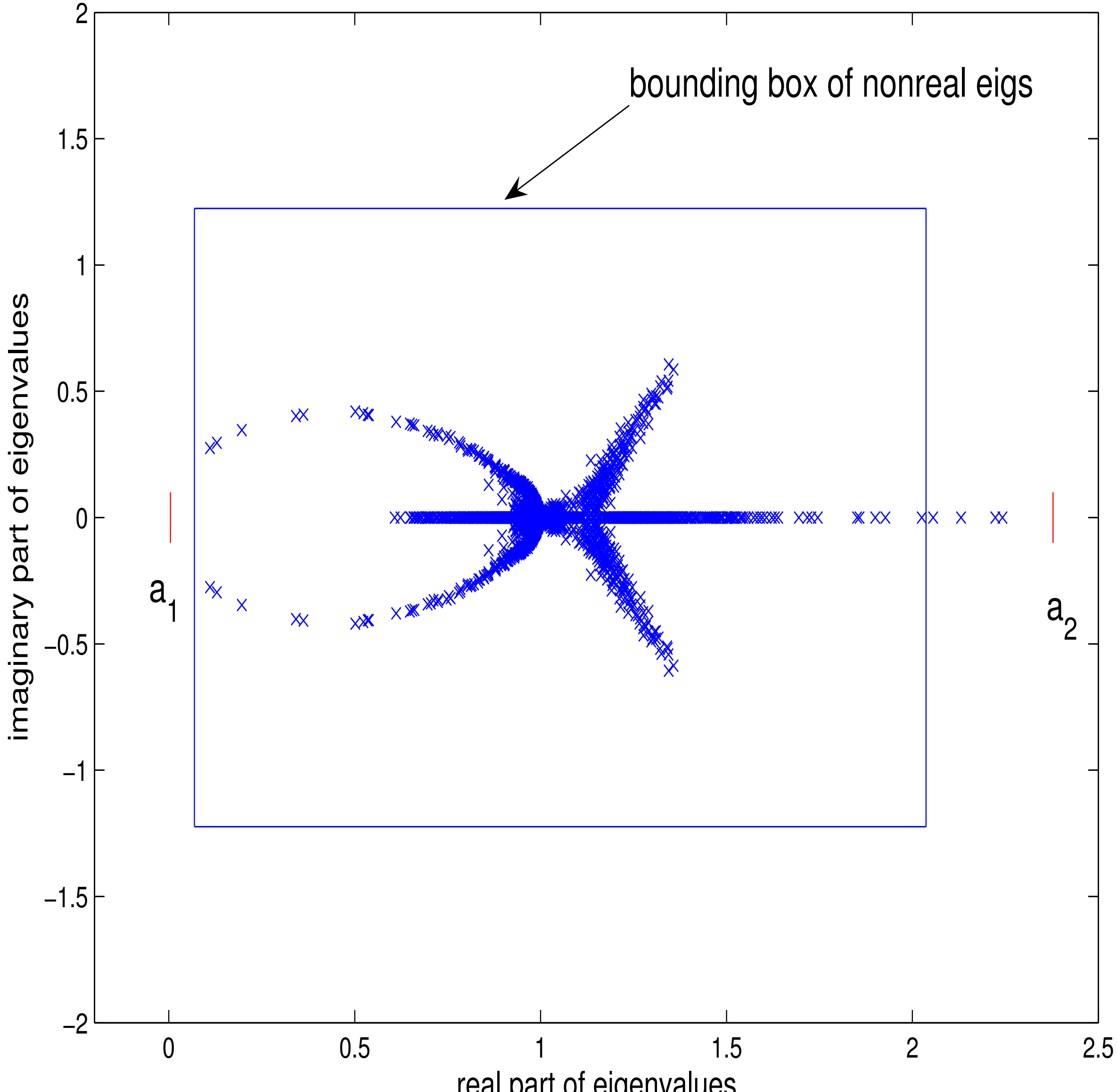
$\star$ If $\Im(\lambda) \neq 0$ then

$$\begin{aligned} \frac{1}{2}(\lambda_{\min}(A) + \lambda_{\min}(\widehat{C})) \leq \quad \Re(\lambda) \quad &\leq \frac{1}{2}(\lambda_{\max}(A) + \lambda_{\max}(\widehat{C})) \\ |\Im(\lambda)| \quad &\leq \sigma_{\max}((I-A)BH^{-\frac{1}{2}}). \end{aligned}$$

$\star$ If $\Im(\lambda) = 0$ then

$$\min\{\lambda_{\min}(A), \lambda_{\min}(\widehat{C})\} \leq \lambda \leq \max\{\lambda_{\max}(A), \lambda_{\max}(\widehat{C})\}$$

Spectral bounds



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- Visit `http://www.dm.unibo.it/~simoncin`